On a nonlinear Dirac equation and constant mean curvature surfaces

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Plan for the talk

- The classical Dirac operator,
  Example: Riemann surfaces
- The conformally invariant functional, definition and results
- Application to spectral theory
- Application to surface theory
- Sketch of proofs
- The estimate (+) in various cases (work in Progress with E. Humbert and B. Morel)
- Example: torus T^2

The classical Dirac operator

Let (M^n, g) be a closed oriented compact Riemannian manifold that carries a spin structure χ. A spin structure is a pair $\chi = (PM, \vartheta)$ where PM is a principal Spin(n) bundle such that

$$\begin{array}{ccccc} PM \times \mathrm{Spin}(n) & \longrightarrow & PM & & \\ & & & \searrow & \\ \downarrow \vartheta \times \Theta & & \downarrow \vartheta & & M \\ & & & \nearrow & \\ \mathrm{SO}(M) \times \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(M) & & \end{array}$$

commutes.

There is a faithful representation

$$\sigma : \mathrm{Spin}(n) \to \mathrm{End}(\mathbb{C}^k), \quad k = 2^{[n/2]}$$

and a bilinear map cl : $\mathbb{R}^n \otimes \mathbb{C}^k \to \mathbb{C}^k$, $X \otimes \varphi \mapsto X \cdot \varphi$ such that

$$(Cl) \qquad X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2g(X, Y)\varphi.$$

Let $S := PM \times_\sigma \mathbb{C}^k$ be the associated bundle. It carries a hermitian metric, a metric connection (induced by the Levi-Civita connection). For any $p \in M$, the map cl induces a parallel multiplication

$$TM \otimes S \to S, \quad X \otimes \varphi \mapsto X \cdot \varphi$$

satisfying the Clifford relations (Cl).

The (classical) Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is then defined as as the first order operator that is locally given by the formula

$$D\varphi = e_i \cdot \nabla_{e_i} \varphi$$

where $e_1, \ldots, e_n$ is a local frame.
D has a self-adjoint extension and is elliptic. Hence its spectrum is real and discrete.

Example: Compact Riemann surfaces

For $n = 2$: $\mathrm{Spin}(2) = \mathrm{SO}(2) = S^1$, $\Theta(z) = z^2$.

$$\sigma(z) = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$$

$$e_1 \cdot = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_2 \cdot = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Choosing a spin structure is the same as choosing a square root of TM, viewed as a complex line bundle.

$$S^+M \otimes_{\mathbb{C}} S^+M = TM \qquad S^-M := \overline{S^+M} = (S^+)^*.$$

$$S := S^+ \oplus S^-$$

$$D = \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}.$$

On a compact Riemann surface of genus γ there are $2^{2\gamma}$ spin structures.

Conformal change of metric

Hitchin, Hijazi

$n \geq 2$. Let $\widetilde{g} = f^2 g$. Then there is a map

$$\begin{array}{rcl} S(M,g,\chi) & \to & S(M,\widetilde{g},\chi) \\ \psi & \mapsto & \end{array}$$

such that

$$|\widetilde{\psi}| = f^{-\frac{n-1}{2}}|\psi| \qquad \text{and} \qquad \widetilde{D}\widetilde{\psi} = f^{-1}\widetilde{D\psi}.$$

The conformally invariant functional

$$\mathcal{F}_q : \Gamma(SM) \setminus \ker D \to \mathbb{R}$$

$$\mathcal{F}_q(\varphi) := \frac{\|D\varphi\|^2_{L^q(M,g)}}{\int \langle D\varphi, \varphi \rangle}$$

With the above identification of spinors this functional is conformally invariant iff
$q = 2n/(n+1)$.

$$\mu_q(M,g,\chi) := \inf\Big\{\mathcal{F}_q(\varphi) \;\Big|\; \varphi \in \Gamma(S) \setminus \ker D, \int \langle D\varphi, \varphi \rangle > 0\Big\}.$$

For $q \geq \frac{2n}{n+1}$ we have a Sobolev embedding

$$L^q_1 \hookrightarrow L^2_{1/2}.$$

This implies that $\mu_q(M,g,\chi) > 0$ for $q \geq \frac{2n}{n+1}$.

Lemma 1. *If* $q > \frac{2n}{n+1}$, *then there is a* $\varphi \in \Gamma_{C^{1,\alpha}}(S)$ *with*

$$\mu_q(M, g, \chi) = \mathcal{F}_q(\varphi).$$

Theorem 2 (A. 2003). *If* $q = \frac{2n}{n+1}$ *and*

$$(+) \qquad \mu_q(M, g, \chi) < \mu_q(S^n) \left(= \frac{n}{2} \omega_n^{1/n}\right),$$

then there is $\varphi \in \Gamma(S)$ *with*

$$\mu_q(M, g, \chi) = \mathcal{F}_q(\varphi).$$

Furthermore, φ *is* $C^{1,\alpha}$ *and* φ *is smooth on* $M \setminus \varphi^{-1}(0)$.
If $n = 2$, *then* φ *is smooth on* M.

Application to Spectral theory

Spectrum of D

$$\dots \leq \lambda_2^- \leq \lambda_1^- < \underbrace{0 = 0 \dots = 0}_{\text{dim ker } D} < \lambda_1^+ \leq \lambda_2^+ \leq \dots$$

λ_1^+ depends on M, g, χ.

Goal

$$\lambda_1^+ \geq \text{nice geometric data}$$

Examples

Friedrich (1980):

Any eigenvalue λ satifies

$$\lambda^2 \geq \frac{n}{4(n-1)} \min \mathrm{scal}.$$

Equality is attained iff M carries a Killing spinor, e.g. $M = S^n$.

Kirchberg (1986 and 1988):

$n = 4k-2$, $Hol \subset U(2k-1)$ or

$n = 4k$, $Hol \subset U(2k)$

$$\lambda^2 \geq \frac{1}{4} \frac{2k}{2k-1} \min \mathrm{scal}$$

Equality e.g. for $M = \mathbb{C}P^{2k-1}$ and $M = \mathbb{C}P^{2k-1} \times T^2$.

Kramer, Semmelmann, Weingart (1997):

$n = 4k$, $Hol \subset \mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$

$$\lambda^2 \geq \frac{1}{4} \frac{m+3}{m+2} \mathrm{scal}.$$

Hijazi (1986):

If $n \geq 3$, then

$$\lambda^2 \operatorname{vol}(M, g)^{2/n} \geq Y(M, [g])$$

where $Y(M, [g])$ is the Yamabe invariant.

Equality holds for S^n.

Bär (1992):

For any metric g on S^2

$$\lambda^2 \operatorname{area}(S^2, g) \geq 4\pi$$

and equality is attained for the round sphere.

New estimates

For finding new estimates one has to find conditions such that λ_1^+ is bounded away from 0.

Corollary 3 (of Theorem 1).

Any metric $\tilde{g} \in [g]$ satisfies

$$\lambda_1^+(M, \tilde{g}, \chi)\mathrm{vol}(M, \tilde{g})^{\frac{1}{n}} \geq \mu_{2n/(n+1)}(M, g, \chi) > 0$$

and equality is attained for a metric (possibly with singularities).

This corollary is due to Lott (1986) for $\ker D = \{0\}$, and due to Amm. (2000) in general.

Application to cmc surfaces

Euler-Lagrange equation of $\mathcal{F}_q$

$$\Leftrightarrow \qquad D\varphi = const \, |\varphi|^{p-2}\varphi$$

where $1/q + 1/p = 1$.

Now $n = 2$, $D\varphi = H \, |\varphi|^2\varphi$.
We write $\varphi = (\varphi_+, \varphi_-)$ and define

$$\alpha := \begin{pmatrix} \mathrm{Re}(\varphi_+ \otimes \varphi_+ - \varphi_- \otimes \varphi_-) \\ \mathrm{Im}(\varphi_+ \otimes \varphi_+ - \varphi_- \otimes \varphi_-) \\ 2\mathrm{Re}(\varphi_+ \otimes \overline{\varphi_-}) \end{pmatrix}.$$

One obtains:
(1) $d\alpha = 0$. Hence there is $F : \widetilde{M} \to \mathbb{R}^3$ such that $\alpha = dF$ and there is a periodicity map $P : \pi_1(M) \to \mathbb{R}^3$ such that

$$F(p \cdot \gamma) = F(p) + P(\gamma) \quad \forall p \in M, \; \gamma \in \pi_1(M)$$

(2) F is a conformal map with odd order branching points

$$|dF| = |\alpha| = |\varphi|^2$$

(3) $F(M)$ has mean curvature H.

$$\left\{\begin{array}{l}\text{Solutions to} \\ D\varphi = const\ |\varphi|^2\varphi \\ \text{on } M\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{Odd-branched conformal} \\ \text{periodic cmc immersions} \\ \text{of } \widetilde{M} \text{ into } \mathbb{R}^3\ (S^3,\ H^3)\end{array}\right\}$$

The estimate (+) in particular cases

Work in progress, Collaboration with E. Humbert, Nancy and B. Morel, Nancy

Now always $q = \frac{2n}{n+1}$.

Question: Which Riemannian spin manifolds satisfy

$$(+) \qquad \mu_q(M,g,\chi) < \mu_q(S^n)?$$

Proposition 4.

$$\mu_q(M,g,\chi) \le \mu_q(S^n)$$

for any Riemannian spin manifold (M,g,χ).

For proving the strict inequality (+) one needs a test spinor $\varphi \in \Gamma(S)$ such that

$$\mathcal{F}_q(\varphi) < \mu_q(S^n).$$

Relations to the Yamabe problem

Note, that if (+) holds, then

$$Y(M,g) \overset{Hijazi}{\leq} \mu_q(M,g,\chi)^2 < \mu_q(S^n)^2 = Y(S^n).$$

Theorem 5 (Yamabe, Trudinger, Aubin, Schoen, Yau 1968–1990).

For any compact (M,g) Riemannian manifold not conformal to S^n the inequality

$$Y(M,g) < Y(S^n)$$

holds.

The theorem solves the famous Yamabe problem: any metric on a compact manifold is conformal to a metric of constant scalar curvature.

The proof in the general case is quite involved (see Lee-Parker). For spin manifolds Witten found a simpler proof, written up in detail in Parker-Taubes, using weighted Sobolev-space theory on asymtotically euclidean spaces. In A.-Humbert, we simplified this proof considerably using only standard analysis on compact manifolds.

Non conformally flat manifolds

Theorem 6 (AHM 2003). *If M is not conformally flat, and if the dimension of M is ≥ 7, then $(+)$ holds.*

The proofs combines

- ideas of the Aubin's proof of Theorem 5 for these cases,
- some results by Bourguignon-Gauduchon,
- new material and
- many calculations in which many terms vanish in the limit.

Conformally flat manifolds

Now, let (M,g) be conformally flat, χ a spin structure.

For $p \in M$ choose a metric $\tilde{g} \in [g]$ that is flat in a neighborhood of p, take normal coordinates, defined on an open set U.

Let G be the Green function for D, i.e. $G(x,y) \in \mathrm{Hom}(S_yM, S_xM)$ is defined for $x,y \in M$, $x \neq y$ depends smoothly on x and y, for any $\varphi \in S_yM$

$$D(\underbrace{G(\,\cdot\,,y)\varphi}_{\in \Gamma(S)}) = \delta(x,y)\varphi$$

$$G(\,\cdot\,,y)\varphi \perp \ker D$$

Expansion in above coordinates yields

$$G(x,y)\varphi = \frac{1}{(n-1)\,\omega_{n-1}}\,\frac{x-y}{|x-y|^n}\cdot\varphi + \beta(x,y)\varphi$$

where $\beta(x,y) \in \mathrm{Hom}(S_yM, S_xM)$ is smooth on $U \times U$.

Definition. The *mass endomorphism* is defined as

$$m_x := \beta(x,x).$$

m_x is selfadjoint, smooth in x. If $n = \dim M$ is even, then the spectrum of m_x is symmetric.

Examples:
$m_x \neq 0$ on $\mathbb{R}P^{4k+3}$,
$m_x = 0$ on flat tori T^n.

Theorem 7 (AHM 2003). *If n is even and if*

$$m \not\equiv 0 \qquad \textit{or} \qquad \ker D \neq \{0\},$$

then $(+)$ holds.

Remark. Until now all statements for λ_1^+ also hold for $|\lambda_1^-|$.

Theorem 8 (AHM 2003). *If n is odd and if*

$$m \not\equiv 0 \qquad \textit{or} \qquad \ker D \neq \{0\},$$

then $(+)$ or $(-)$ holds.

Here $(-)$ is $(+)$ with λ_1^+ replaced by λ_1^-.

Any compact Riemann surface of genus ≥ 1 has a spin structure with non-vanishing α-genus ($n \equiv 2 \bmod 8$ generalization of $\widehat{A}(M)$). Hence, for this spin structure $\ker D \neq \{0\}$.

Corollary 9. *Any compact Riemann surface M of genus ≥ 1 has a spin structure such that $(+)$ holds. As a consequence we obtain a periodic odd-branched conformal cmc immersion of $\widetilde{M}$ into $\mathbb{R}^3$ with*

$$H^2 \,\mathrm{area}(F(\widetilde{M})/P(\pi_1)) < 4\pi.$$

Example: The torus T^2

Let T^2 carry an arbitrary conformal structure. It has 3 spin structures with $\alpha(T^2,\chi) = 0$, and one with $\alpha(T^2,\chi) \neq 0$ (this is the trivial covering of SO(T^2)).

One obtains solution to $D\varphi = \mu|\varphi|^2\varphi$ and conformal cmc immersion of $\mathbb{R}^2$ such that

$$\int_{T^2} H^2 = \int \mu^2|\varphi|^4 = \mu_q(T^2)^2 < 4\pi.$$

Lemma 10. *There are no branching points, i.e.* $\varphi(x) \neq 0 \ \forall x \in T^2$.

Proof. Applying Gauss-Bonnet to $(T^2 \setminus \varphi^{-1}(0), |\varphi|^4 g_{eucl})$ yields

$$4\pi \cdot \#\{\text{branch points}\} \ \leq \ \int K = \int \kappa_1\kappa_2$$

$$\leq \ \int \left(\frac{\kappa_1+\kappa_2}{2}\right)^2 = \int H^2 < 4\pi$$

Conclusions for T^2

Let χ be the *trivial* spin structure on T^2 ($\Leftrightarrow \ker D \neq \{0\}$). Fix a conformal class $[g]$. We assume g is flat, i.e. (T^2, g) is isometric to $\mathbb{R}^2/\Gamma$.

- Then the infimum

$$\mu = \inf_{\tilde{g} \in [g]} \lambda_1^+(T^2, \tilde{g}, \chi) \sqrt{\mathrm{area}(T^2, \tilde{g})}$$

  is positive and smaller than $\sqrt{4\pi}$. The infimum is attained by a smooth metric without singularities.

- For many conformal structures (e.g. the square torus) the infimum is not attained by a flat metric.

- There is a conformal map $F : \mathbb{R}^2 \to \mathbb{R}^3$ and a homomorphism $P : \Gamma \to \mathbb{R}^3$ such that

$$F(x + \gamma) = F(x) + P(\gamma),$$

  such that $F(\mathbb{R}^2)$ has constant mean curvature μ, and the area of a fundamental domain is 1.

- Similar immersions exist into S^3 and H^3.

Addtional Information

On this page we want to add some references for readers of the internet version of these slides:

The proof of Lemma 1 and Theorem 2 is contained in [1] and [2]. There you will also find further references, in particular to the spinorial Weierstrass representation, which underlies the application to cmc surfaces.

For a good overview article over the Yamabe problem, we refer to [3], the simplification for spin manifolds is explained in [4].

Theorem 6 is proved in [5]. The results about conformally flat manifolds will appear in [6].

References

[1] Bernd Ammann, A variational problem in conformal spin geometry, Habilitationsschrift Universitaet Hamburg, 2003

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[3] Lee, Parker, The Yamabe problem, Bull AMS, New series 17 (1987) 37–91

[4] Ammann, Humbert, On a nonlinear Dirac equation of Yamabe type, Preprint math.DG/0304043

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[6] Ammann, Humbert, Morel, preprint in preparation.

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