What is Ramsey Theory?

It might be described as the study of
unavoidable regularity in large structures.

Complete disorder is impossible.

T. Motzkin

Ramsey's Theorem (1930)

For any $k < l$ and r, there exists $R = R(k,l,r)$ so that for any r-coloring of the k-element sets of an R-element set, there is always some l-element set with all of its k-element subsets having the same color.

Frank Plumpton Ramsey

(1903-1930)

Euclidean Ramsey Theory

$X \subset \mathbf{E}^k$ - finite

Cong(X) - family of all $X' \subset \mathbf{E}^k$ which are congruent to X

(i.e., "copies" of X up to some Euclidean motion)

X is said to be Ramsey if for all r there exists

N = N(X,r) such that for every partition $\mathbf{E}^N = C_1 \cup C_2 \cup \ldots \cup C_r$,

we have $X' \in C_i$ for some $X' \in$ Cong(X) and some i.

$$\mathbf{E}^N \xrightarrow{r} X$$

Compactness Principle

If $\mathbf{E}^N \xrightarrow{r} X$ then there is a finite subset $Y \in \mathbf{E}^N$

such that $Y \xrightarrow{r} X$

Example

$X =$ 

$|X| = 2$

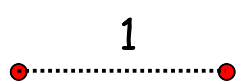
For a given r, take $Y_r \subset \mathbf{E}^r$ to be the r+1 vertices of

a unit simplex in $\mathbf{E}^r$.

Compactness Principle

If $\mathbf{E}^N \xrightarrow{\mathbf{r}} X$ then there is a finite subset $Y \in \mathbf{E}^N$ such that $Y \xrightarrow{\mathbf{r}} X$

Example

$X =$ 

$|X| = 2$

For a given r, take $Y_r \subset \mathbf{E}^r$ to be the r+1 vertices of a unit simplex in $\mathbf{E}^r$. Then $Y_r \xrightarrow{r} X$.

Let Q^n denote the set of 2^n vertices $\{(x_1,\ldots,x_n) : x_k = 0 \text{ or } 1\}$ of the n-cube. Then Q^n is Ramsey.

Theorem. For any k and r, there exists N = N(k,r) such that any r-coloring of Q^N contains a monochromatic $\sqrt{2}\,Q^k$.

Idea of proof: (induction)   k = 1   Choose N(1,r) = r + 1

Consider the r + 1 points:

←— r+1 —→

(1,0,0,……,0)
(0,1,0,……,0)
(0,0,1,……,0)
⋮
(0,0,0,……,1)

Since only r colors are used then some pair must have the same color, say

$$(\ldots\ldots\ldots\ldots,1,\ldots\ldots\ldots\ldots,0,\ldots\ldots\ldots\ldots)$$

and

$$(\ldots\ldots\ldots\ldots,0,\ldots\ldots\ldots\ldots,1,\ldots\ldots\ldots\ldots)$$

This is a monochromatic $\sqrt{2}Q^1$.

So far, so good!

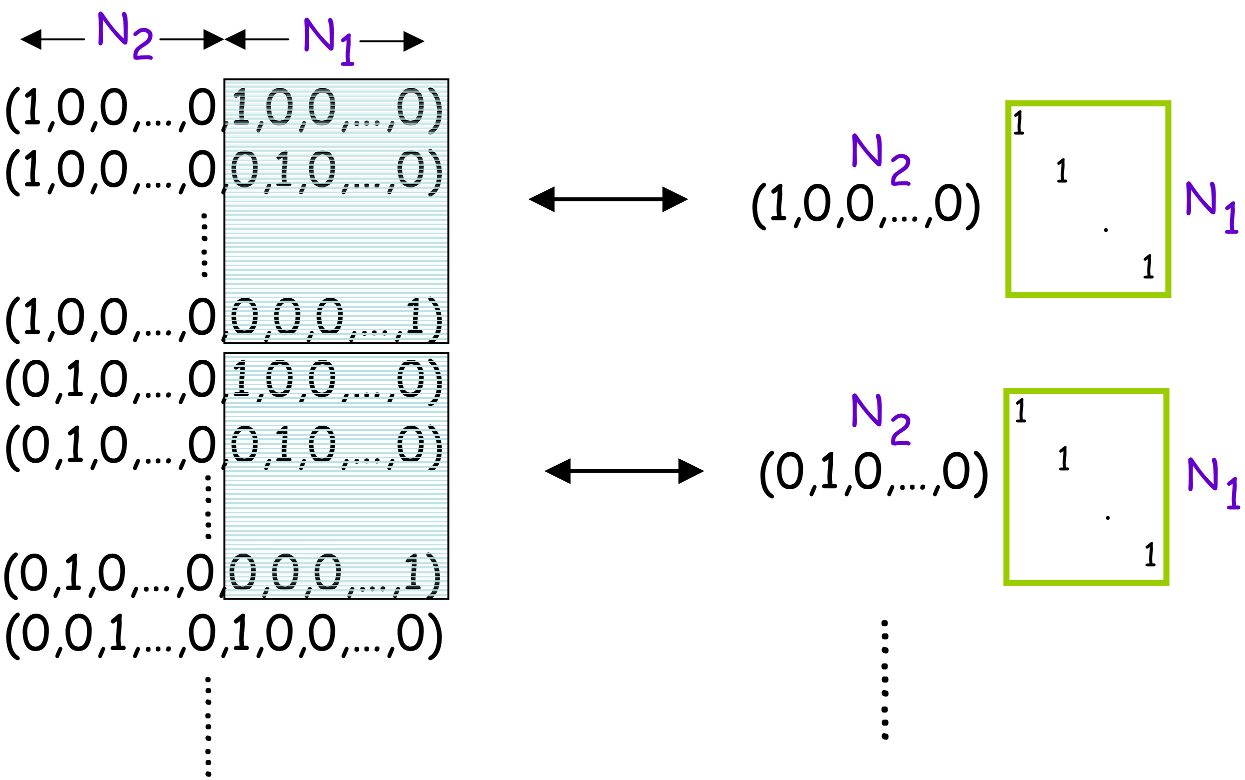
k = 2 Choose $N(2,r) = (r^{r+1} + 1) + (r + 1)$

$= N_2 + N_1$

k = 2 Choose $N(2,r) = (r^{r+1} + 1) + (r + 1)$

$$= N_2 + N_1$$

Consider the $N_2 N_1$ points:



Since the N_2 points represented by the $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cdot & \\ & & & 1 \end{bmatrix}$ N_1's

can be r-colored in at most r^{N_1} ways, then the original r-coloring of $Q^{N_2 + N_1}$ induces an r^{N_1}- coloring of Q^{N_2}.

Since $N_2 = r^{r+1} + 1 = r^{N_1} + 1$, some pair has the same coloring, say

$$\begin{array}{c} \quad i_2 \qquad\qquad j_2 \\ (\ldots\ldots,1,\ldots\ldots 0,\ldots\ldots) \end{array} \begin{array}{c} i_1 \;\; j_1 \\ \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{array} \leftarrow$$

$$(\ldots\ldots,0,\ldots\ldots 1,\ldots\ldots) \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \leftarrow$$

Thus, all 4 are monochromatic

These 4 points form a monochromatic $\sqrt{2}Q^2$:

$$\begin{array}{c} (\ldots,\overset{i_2}{1},\ldots\ldots\overset{j_2}{0},\ldots\ldots,\overset{i_1}{1},\ldots\ldots,\overset{j_1}{0},\ldots\ldots) \\ (\ldots,1,\ldots\ldots 0,\ldots\ldots,0,\ldots\ldots,1,\ldots\ldots) \\ (\ldots,0,\ldots\ldots 1,\ldots\ldots,1,\ldots\ldots,0,\ldots\ldots) \\ (\ldots,0,\ldots\ldots 1,\ldots\ldots,0,\ldots\ldots,1,\ldots\ldots) \end{array}$$

For k = 3, we can take $N(3,r) = N_3 + N_2 + N_1$

where $$N_3 = 1 + r^{N_2 N_1} = 1 + r^{(1+r^{1+r})(1+r)},\text{etc.}$$

For k = 3, we can take $N(3,r) = N_3 + N_2 + N_1$

where $$N_3 = 1 + r^{N_2 N_1} = 1 + r^{(1+r^{1+r})(1+r)}, \text{etc.}$$

Continuing this way, the theorem is proved. $\blacksquare$

Note that by this technique, the bounds we get are rather large.

For example, it shows that $N(4,2) \leq 2^{2^{7}} + 13$.

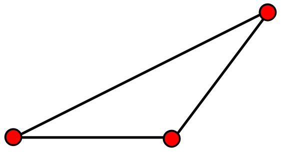
What is the true order of growth here?

With this technique, we can prove the:

Product Theorem. If X and Y are Ramsey then the Cartesian product $X \times Y$ is also Ramsey.

Corollary: (Any subset of) the vertices of an n-dimensional rectangular parallelepiped is Ramsey.

For example, any acute triangle is Ramsey.

What about  ?

How can we get obtuse Ramsey triangles?

Example.

Choose n = R(7, 9, r) and consider the set S of points $\overline{x}$ in $\mathbf{E}^n$

having all coordinates zero except for 7 coordinates which

have in order the values 1, 2, 3, 4, 3, 2, 1.

$$\overline{x} = (0\ 0\ 0\ 1\ 0\ 2\ 0\ 0\ 0\ 3\ 4\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 2\ 0\ 1\ 0\ 0\ 0\ 0)$$

There are $\binom{n}{7}$ such points in S.

Any r-coloring of S induces an r-coloring of the 7-sets of {1,2,……,n}

By the choice of $n = R(7, 9, r)$, there exists some 9-set $\{i_1, i_2, \ldots, i_9\}$ with all its 7-sets having the same color.

$$\bar{x} = (\ldots\ldots x_{i_1} \ldots\ldots x_{i_2} \ldots\ldots x_{i_3} \ldots\ldots x_{i_4} \ldots\ldots x_{i_5} \ldots\ldots x_{i_6} \ldots\ldots x_{i_7} \ldots\ldots x_{i_8} \ldots\ldots x_{i_9} \ldots\ldots)$$

$$A = (\ldots\ldots 1 \ldots\ldots 2 \ldots\ldots 3 \ldots\ldots 4 \ldots\ldots 3 \ldots\ldots 2 \ldots\ldots 1 \ldots\ldots 0 \ldots\ldots 0 \ldots\ldots)$$

$$B = (\ldots\ldots 0 \ldots\ldots 1 \ldots\ldots 2 \ldots\ldots 3 \ldots\ldots 4 \ldots\ldots 3 \ldots\ldots 2 \ldots\ldots 1 \ldots\ldots 0 \ldots\ldots)$$

$$C = (\ldots\ldots 0 \ldots\ldots 0 \ldots\ldots 1 \ldots\ldots 2 \ldots\ldots 3 \ldots\ldots 4 \ldots\ldots 3 \ldots\ldots 2 \ldots\ldots 1 \ldots\ldots)$$

$\mathrm{dist}(A, B) = \sqrt{8}$ $\mathrm{dist}(B, C) = \sqrt{8}$ $\mathrm{dist}(A, C) = \sqrt{26}$

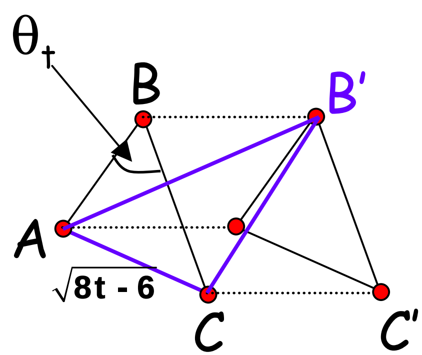
Thus, the $(\sqrt{8}, \sqrt{8}, \sqrt{26})$-triangle is Ramsey.

In general, this technique shows that the triangle with side lengths

$\sqrt{2t}$, $\sqrt{2t}$ and $\sqrt{8t-6}$ is Ramsey.

Note that the angle θ_t between the short sides $\rightarrow 180^{o}$ as $t \rightarrow \infty$.

Form the product:



By the product theorem, triangle AB′C is also Ramsey.

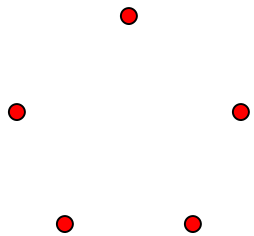
Theorem (Frankl, Rödl) All triangles are Ramsey.

Theorem: (Frankl/Rödl – 1990)

For any (non-degenerate) simplex $S \in \mathbf{E}^k$,

there is a $c = c(S)$ so that

$$\mathbf{E}^{c \log r} \xrightarrow{r} S$$

What about  ?

Theorem (I. Kríz - 1991)

If $X \subset \mathbf{E}^N$ has a transitive solvable group of isometries then X is Ramsey.

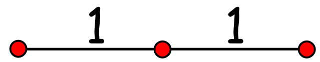
Corollary. The set of vertices of any regular n-gon is Ramsey.

Theorem (I. Kríz - 1991)

If $X \subset \mathbf{E}^N$ has a transitive group of isometries which has a solvable subgroup with at most 2 orbits then X is Ramsey.

Corollary. The set of vertices of any Platonic solid is Ramsey.

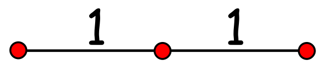
Are there any non-Ramsey sets??

Proof that 

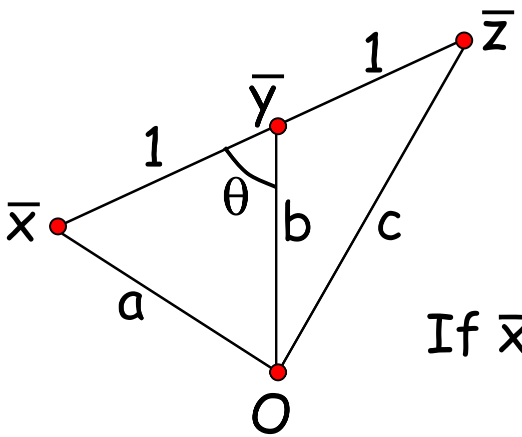
is not Ramsey.

4-color each $\bar{\mathbf{x}} \in \mathbf{E}^N$ according to $\lfloor \bar{\mathbf{x}} \cdot \bar{\mathbf{x}} \rfloor$ (mod 4).

(alternating spherical shells about O with decreasing thickness)

Proof that  is not Ramsey.

4-color each $\bar{x} \in \mathbf{E}^N$ according to $\lfloor \bar{x} \cdot \bar{x} \rfloor \pmod 4$. Then



$$a^2 = b^2 + 1 - 2b\cos\theta$$

$$c^2 = b^2 + 1 + 2b\cos\theta$$

Thus, $$a^2 + c^2 = 2b^2 + 2 \qquad (*)$$

If $\bar{x}$, $\bar{y}$ and $\bar{z}$ have color d, then

$$a^2 = 4k_a + d + \varepsilon_a,\ 0 \le \varepsilon_a < 1$$

$$b^2 = 4k_b + d + \varepsilon_b,\ 0 \le \varepsilon_b < 1$$

$$c^2 = 4k_c + d + \varepsilon_c,\ 0 \le \varepsilon_c < 1$$

By (*), $$4k_a + 4k_c + 2d + \varepsilon_a + \varepsilon_c = 8k_b + 2d + 2\varepsilon_b + 2$$

i.e., $$4M - 2 = 2\varepsilon_b - \varepsilon_a - \varepsilon_c$$

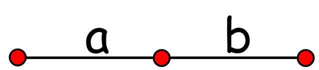
which is impossible since $-2 < 2\varepsilon_b - \varepsilon_a - \varepsilon_c < 2$.

Call X spherical if X is a subset of some sphere $S^d(\rho)$ in $\mathbf{E}^k$

Theorem (Erdős, Graham, Montgomery, Rothschild, Spencer, Straus)

X is Ramsey $\Rightarrow$ X is spherical.

Corollary.

X = 

(collinear) is not Ramsey.

In fact, $\mathbf{E}^N \not\xrightarrow{16} X$ for any N.

Is 16 best possible??

Definition: X is called sphere-Ramsey if for all r, there exist $N = N(X,r)$ and $\rho = \rho(X,r)$ such that for all partitions $S^N(\rho) = C_1 \cup C_2 \cup \ldots \cup C_r$, some C_i contains a copy of X.

Note: sphere-Ramsey $\Longrightarrow$ Ramsey $\Longrightarrow$ spherical

Theorem (Matoušek/Rödl)

If $X \subset S^d(1)$ is a simplex then for all r and all $\varepsilon > 0$,

there exists $N = N(X, r, \varepsilon)$ such that

$$S^N(1+\varepsilon) \xrightarrow{r} X$$

Thus, X is sphere-Ramsey.

Is the ε really needed? Yes !

Theorem (RLG)

Suppose $X = \{\overline{x}_1, \ldots, \overline{x}_k\} \subset S^d(1)$ is unit-sphere-Ramsey

(i.e., $S^N(1) \xrightarrow{r} X$, $N = N(X,r)$)

Then for any linear dependence $\sum_{i \in I} c_i \overline{x}_i = \overline{0}$,

there must exist a nonempty set $J \subseteq I$ with $\sum_{j \in J} c_j = 0$.

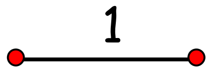
Corollary. If X above has $\overline{0} \in \text{conv}(X)$ then X is not unit-sphere-Ramsey.

(since $\overline{0} = \sum_{i \in I} c_i \overline{x}_i$ with all $c_i > 0$).

Suppose that we fix the dimension of the space $\mathbf{E}^n$.

What is true in this case?

The **simplest** set:



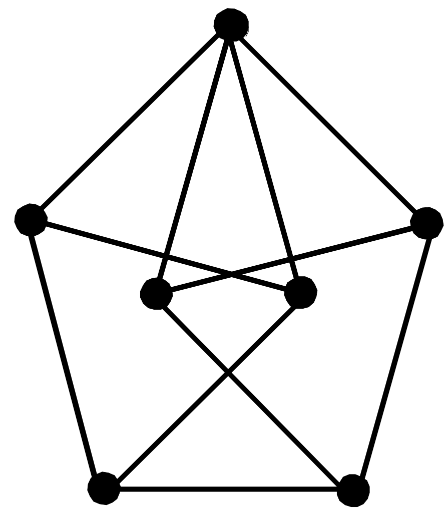
Define $\chi(\mathbf{E}^2)$, the chromatic number of $\mathbf{E}^2$, to be

the least r such for some r-coloring $\mathbf{E}^2 = C_1 \cup C_2 \cup \ldots \cup C_r$,

no C_i contains 2 points at a distance of 1 from each other.

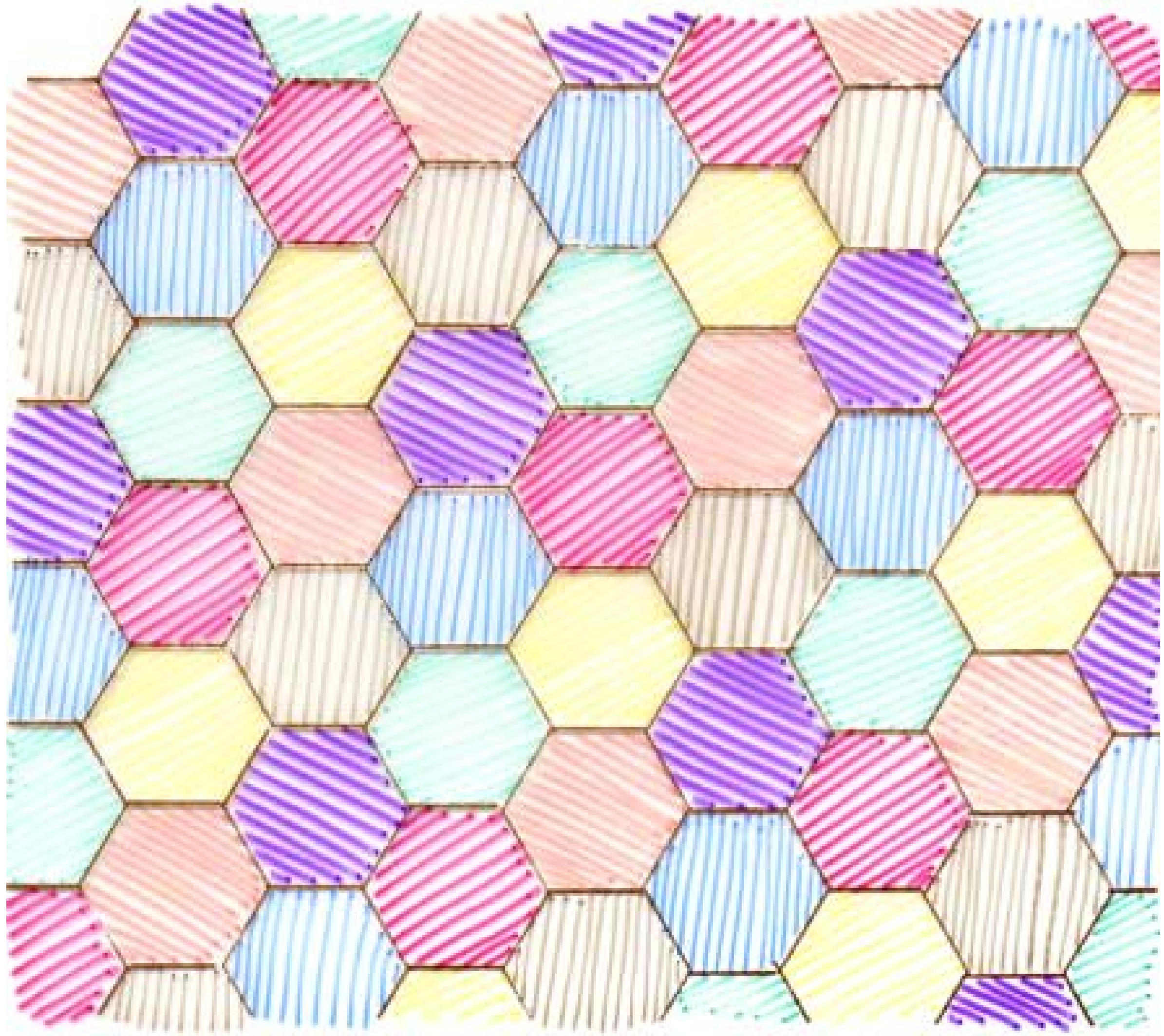
In other words, no unit distance occurs monochromatically

What is the value of $\chi(\mathbf{E}^2)$?? $4 \leq \chi(\mathbf{E}^2) \leq 7$



Mosers' graph M

$$\chi(\mathbf{E}^2) \geq \chi(M) = 4$$



$\chi(\mathbf{E}^2) \leq 7$

Define $\chi(\mathbf{E}^2)$, the chromatic number of $\mathbf{E}^2$, to be

the least r such for some r-coloring $\mathbf{E}^2 = C_1 \cup C_2 \cup .... \cup C_r$,

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In other words, no unit distance occurs monochromatically

What is the value of $\chi(\mathbf{E}^2)$?? $4 \leq \chi(\mathbf{E}^2) \leq 7$

$$6 \leq \chi(\mathbf{E}^3) \leq 15$$

Nechustan (2000) Radoičić/Tóth (2002)

Define $\chi(\mathbf{E}^2)$, the chromatic number of $\mathbf{E}^2$, to be

the least r such for some r-coloring $\mathbf{E}^2 = C_1 \cup C_2 \cup .... \cup C_r$,

no C_i contains 2 points at a distance of 1 from each other.

In other words, no unit distance occurs monochromatically

What is the value of $\chi(\mathbf{E}^2)$?? $4 \leq \chi(\mathbf{E}^2) \leq 7$

For $\mathbf{E}^n$ it is known that:

$$(1 + o(1))(\frac{6}{5})^n \leq \chi(\mathbf{E}^n) \leq (3 + o(1))^n$$

Theorem (O'Donnell – 2000)

For every *g*, there is a 4-chromatic unit distance graph *G* in $\mathbf{E}^2$ having girth greater than *g*.

This is perhaps evidence supporting the conjecture that:

$$\chi(\mathbf{E}^2) \overset{?}{\geq} 5$$

Problem: ($1000) Determine the value of $\chi(\mathbf{E}^2)$.

A little set theory:

Most of us work in ZFC, that is, the usual Zermelo-Fraenkel axioms together with the Axiom of Choice:

AC: Every family F of nonempty sets has a choice function, i.e., there is a function f such that f(S) ε S for every S in F

A weaker form of **AC** is **DC**, the principle of dependent choices:

DC: If E is a binary relation on a nonempty set *A*, and for every a ε *A*, there exists b ε B with aEb, then there is a sequence $a_1, a_2, \dots, a_n, \dots$ such that $a_n E a_{n+1}$ for every $n < \omega$.

Another useful axiom in set theory is:

LM: Every set of real numbers is Lebesgue measurable.

Theorem (Solovay – 1970):

Assuming the existence of an inaccessible cardinal, the system of axioms **ZF** + **DC** + **LM** is consistent.

Theorem (Shelah-Soifer – 2003):

Assume that any finite unit distance plane graph has chromatic number not exceeding 4. Then:

(i) In **ZFC** the chromatic number of the plane is 4;

(ii) In **ZF** + **DC** + **LM** the chromatic number of the plane is 5, 6 or 7.

The beginnings

(E.Klein)

Any set X of 5 points in the plane in *general* position must contain the vertices of a convex 4-gon.

For each n, let f(n) denote the least integer so that any set X of f(n) points in the plane in general position must contain the vertices of a convex n-gon.

Does f(n) always exist?

If so, determine or estimate it.

Erdős and Szekeres showed that f(n) always exists and, in fact,

$$2^{n-2}+1 \le f(n) \le \binom{2n-4}{n-2}+1$$

Erdős and Szekeres showed that f(n) always exists and, in fact,

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} + 1$$

They gave several proofs that f(n) exists, one of which used their independent discovery of **Ramsey's Theorem**.

$$2^{n-2}+1 \le f(n) \le \binom{2n-4}{n-2}+1$$

$$f(n) \le \binom{2n-4}{n-2}$$ Chung/Graham (1997)

$$f(n) \le \binom{2n-4}{n-2}-2n+7$$ Kleitman/Pachter (1997)

$$f(n) \le \binom{2n-5}{n-2}+2$$ G. Tóth/Valtr (1997)

Conjecture ($1000)

$$f(n) = 2^{n-2} + 1, \quad \text{for } n \geq 2$$

More beginnings

van der Waerden's Theorem (1927)

In any partition of $\mathbf{N} = \{1,2,3,......\}$ in finitely many classes $C_1 \cup C_2 \cup ... \cup C_r$, some C_i must contain k-term arithmetic progressions for all k.

k-AP

Erdös and Turán ask in 1936 which C_i has k-AP's ?

They conjectured that if C_i is "dense enough" then this should imply that C_i has k-AP's.

Define $r_k(n)$ to be the least integer such that any set $X \subseteq \{1,2,\ldots,n\}$ with $|X| \geq r_k(n)$ must contain a k-AP.

Erdős and Turán conjectured that $r_k(n) = o(n)$.

Progress was slow

$r_3(n) \geq n \exp\left(-c\sqrt{\log n}\right)$ Behrend (1946)

$r_3(n) = O\left(n/(\log\log n)^c \right)$ Roth (1954)

$r_4(n) = o(n)$ Szemerédi (1969)

$r_k(n) = o(n)$ for all k Szemerédi (1974)

($1000 and the **regularity lemma**)

Progress is now *accelerating*

$r_3(n) \geq n \exp\left(-c\sqrt{\log n}\right)$ Behrend (1946)

$r_3(n) = O\left(n / (\log\log n)^c \right)$ Roth (1954)

$r_4(n) = o(n)$ Szemerédi (1969)

$r_k(n) = o(n)$ for all k Szemerédi (1974)

$r_3(n) = O\left(\frac{n}{(\log n)^{1/3}}\right)$ Heath-Brown (1987), Szemerédi (1990)

$r_4(n) = O\left(n / (\log\log n)^c \right)$ Gowers (1998)

$r_k(n) = O\left(n / (\log\log n)^{c_k} \right)$ Gowers (2000)

Define W(n) to be the least integer W (by van der Waerden) so that every 2-coloring of {1,2,...,W} has an n-AP in one color.

<u>Corollary</u> (*Gowers* 2000)

($1000)

$$W(n) \leq 2^{2^{2^{2^{2^{n+9}}}}}, \text{ for all } n.$$

Define W(n) to be the least integer W (by van der Waerden) so that every 2-coloring of {1,2,...,W} has an n-AP.

Corollary (Gowers 2000)

$$W(n) \leq 2^{2^{2^{2^{2^{n+9}}}}}, \text{ for all } n.$$

Conjecture ($1000): $W(n) \leq 2^{n^2}$ for all n.

Best current lower bound is $W(n+1) \geq n \cdot 2^n$, n prime (Berlekamp 1968)

What can be true for partitions of $\mathbf{E}^2$ if we allow an arbitrary finite number of colors?

What can be true for partitions of $\mathbf{E}^2$ if we allow an arbitrary finite number of colors?

Theorem. (RLG) For every r, there exists a least integer T(r) so that for any partition of $\mathbf{Z}^2 = C_1 \cup C_2 \cup \ldots \cup C_r$, some C_i contains the vertices of a triangle of area exactly T(r).

How large is T(r)?

It can be shown that $T(r) > (\frac{1}{2})\,\text{l.c.m}\,(2,3,\ldots,r) = e^{(1+o(1))r}$.

The best known upper bound grows much faster than the (infamous) van der Waerden function W.

For example, let W(k,r) denote the least value W so that in any r-coloring of the first W integers, there is always formed a monochromatic k-term arithmetic progression.

Then $T(3) < 725760 \cdot 1725761!\ W(725761!+1,3)!$

Actually, T(3) = 3.

What is the truth here??

What if you allow **infinitely many** colors?

Theorem (Kunen)

Assuming the Continuum Hypothesis, it is possible to partition $\mathbf{E}^2$ into countably many sets, none of which contains the vertices of a triangle with rational area.

Theorem (Erdős/Komjáth)

The existence of a partition of $\mathbf{E}^2$ into countably many sets, none of which contains the vertices of a **right** triangle is equivalent to the Continuum Hypothesis.

Edge-Ramsey Configurations

A finite configuration L of line segments in $\mathbf{E}^k$ is said to be edge-Ramsey if for any r there is an N = N(L,r) so that in any r-coloring of the line segments in $\mathbf{E}^N$, there is always a monochromatic copy of L.

Edge-Ramsey Configurations

A finite configuration L of line segments in $\mathbf{E}^k$ is said to be edge-Ramsey if for any r there is an N = N(L,r) so that in any r-coloring of the line segments in $\mathbf{E}^N$, there is always a monochromatic copy of L.

What do we know about edge-Ramsey configurations?

Theorem (EGMRSS)

If L is edge-Ramsey then all the edges of L must have the same length.

Theorem (RLG)

If L is edge-Ramsey then the endpoints of the edges of E must lie on two spheres.

Theorem (RLG)

If the endpoints of the edges of L do not lie on a sphere and the graph formed by L is not bipartite then L is not edge-Ramsey.

Theorem (Cantwell)

The edge set of an n-cube is edge-Ramsey.

Theorem (Cantwell)

The edge set of a regular n-gon is not edge-Ramsey if $n = 5$ or $n > 6$.

Question: Is the edge set of a regular hexagon edge-Ramsey?

(Big) Problem: Characterize edge-Ramsey configurations.

There is currently no plausible conjecture.

We know:

sphere-Ramsey $\Longrightarrow$ Ramsey $\Longrightarrow$ spherical $\Longrightarrow$ rectangular

What about the **other direction?**

sphere-Ramsey $\overset{?}{\underset{\$1000}{\Longleftarrow}}$ Ramsey $\overset{?}{\underset{\$1000}{\Longleftarrow}}$ spherical

I'll close with some easier(?) problems:

Question: What are the unit-sphere-Ramsey configurations?

Conjecture ($50)

For any triangle T, there is a 3-coloring of $\mathbf{E}^2$ with no monochromatic copy of T.

Conjecture ($100):

Every 2-coloring of $\mathbf{E}^2$ contains a monochromatic copy of every triangle, except possibly for a single equilateral triangle.

Conjecture ($100)

Any 4-point subset of a circle is Ramsey.

Conjecture ($1000)

Every spherical set is Ramsey.