P polytope (simplicial)

A missing face or empty face is a subset F of vertices that to not form a face but every proper subset of F form a face

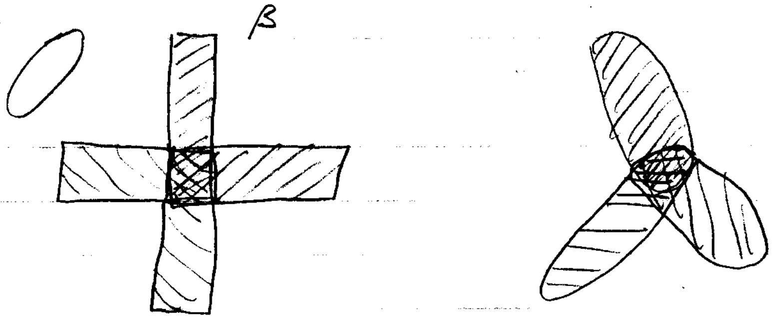
$\mathrm{Skel}_r(P) = \{F \text{ face of } P : \dim F \leq r\}$

THM (Perles)
\# combinatorial types of $\mathrm{Skel}_r(P)$ where P is a d-polytope with $d+\beta$ vertices is bounded for fixed β and r.

follows from

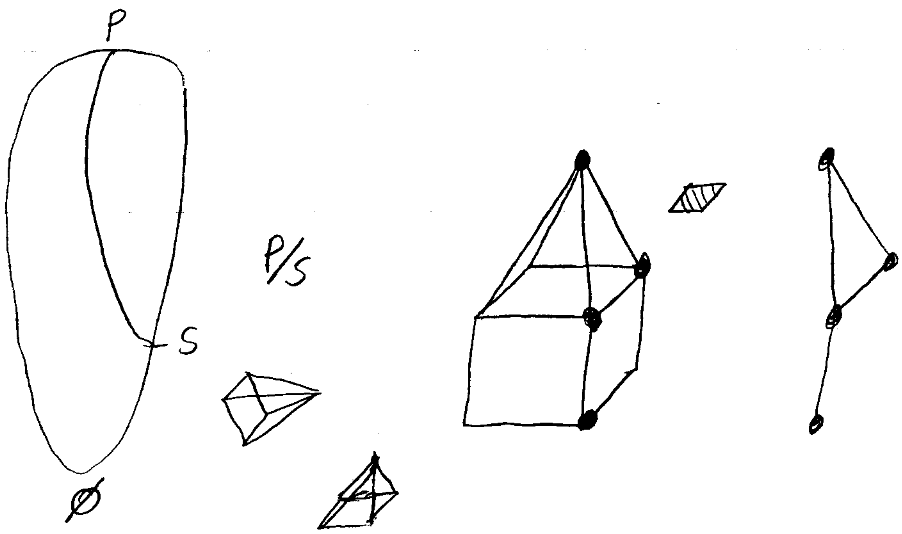
THM: The number of missing faces of dimension $\leq r$ is a d-polytope with $d+\beta$ vertices is bounded by a function of β and r.

$\boxed{f_0(Q) - \dim Q > \beta}$



$F_1 \; F_2 \ldots F_\beta$ are sunflower of missing faces

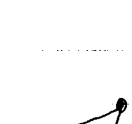
$F_1 \cap F_2 \cap \cdots F_m = S$ is a face $\qquad P/S = Q$



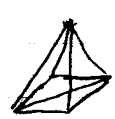
P d-polytope $\quad f_0(P) \, f_1(P) \cdots f_{d-1}(P)$ are given what is max # of missing r-dim faces ?

THM: (Migliore & Nagel)

<u>Conj</u> maximum attained for the Billera–Lee polytopes



$$\sum_{|S|=j} \beta_i(P[S])$$



$j = r+1 \quad i = r-1 \quad$ missing r-face

Question: Are there only finitly many projectively unique d-polytopes

For general d-polytopes $Skel_{d-2}(P)$ determines P

THM (Perles)
For simplicial polytopes $Skel_{\lfloor \frac{d}{2} \rfloor}(P)$ determines P

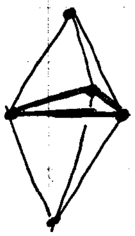
THM (Mum-Blind) For simple polytopes $G(P)$ ($Skel_1(P)$) determines P.

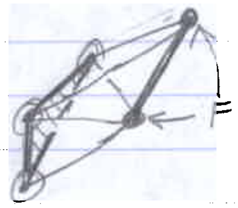
$\underline{d=4}$ P a simplicial polytope

F is a set of vertices of P. $|F| = k+1$ and every proper subset of F is a face of P

F is a k-face $\iff$ $Skel_{d-k-1}(ast(F,P))$ is contractible in $Skel_{d-k}(ast(F,P))$

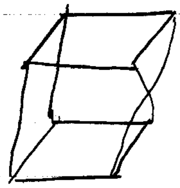
$ast(F,P) = \{S \text{ face of } P : S \cap F = \emptyset\}$





$d=3$
$k=1$

G graph of simple d-polytope
G is d-regular graph



We will consider orientation of the graph which are 1) acyclic
2) good = in every face of the polytope there is a unique sink

Let $\mathcal{O}$ be an acyclic orientation of G
for a vertex $v \in G$ let $\deg(v)$ be the number
of edges oriented toward v.

$h_k^{\mathcal{O}} = \#$ vertices of degree k

$$X = h_0^{\mathcal{O}} + 2h_1^{\mathcal{O}} + 4h_2^{\mathcal{O}} + \cdots + 2^d h_d^{\mathcal{O}}$$

<u>Claim</u>: if the orientation is good then
X is precisely the number of nonempty
faces of P

If the orientation is bad then
$X > \#$ of non-empty faces of P