Forbidden Families of Geometric Permutations

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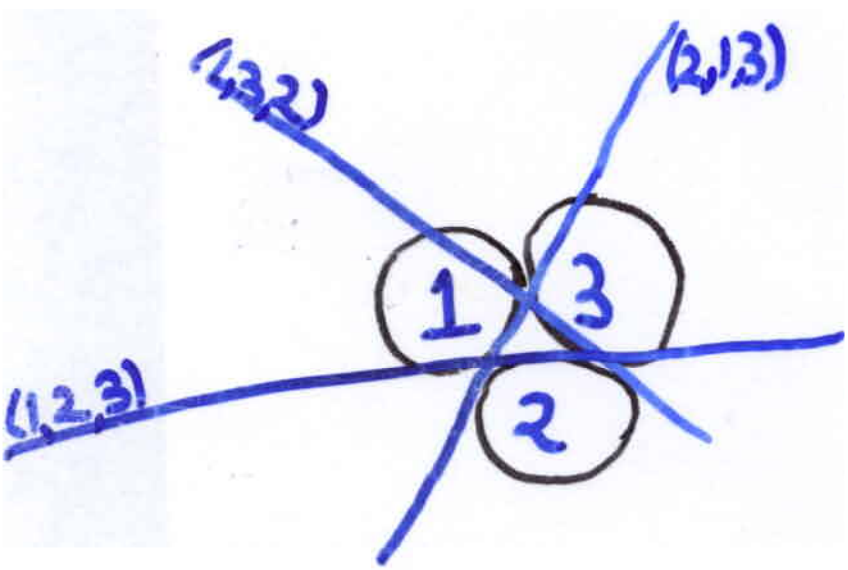
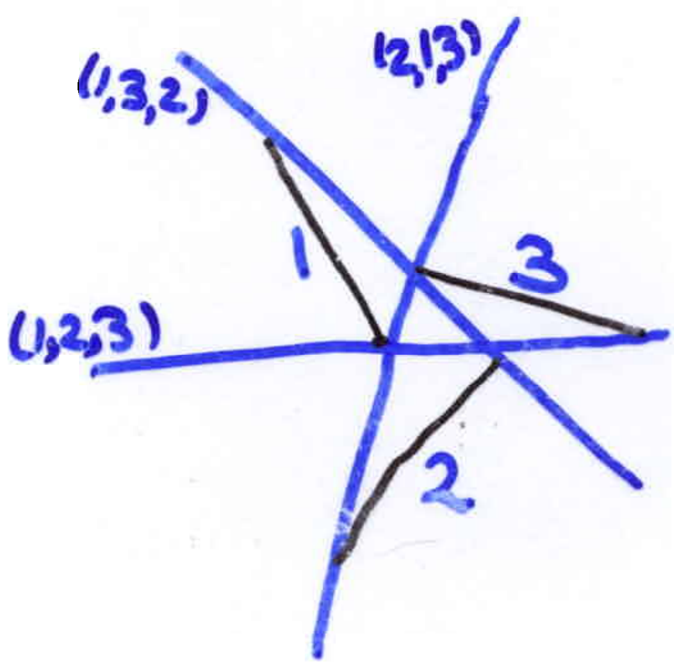
A geometric permutation induced by a transversal line for a finite family of disjoint convex sets in R^d is the order and its reverse in which the transversal meets the members of the family.





Motivation:

1) Computational Geometry: Visibility problems, ...
2) Helly type problems on line transversals.

Ex 1. (Tverberg): For disjoint translates of a convex set, all in the plane, if any 5 admit a transversal then all admit a transversal.

(transversal = line thru all the sets

) (Also true for convex sets of diameter ≤ 1 and area $\geq \alpha > 0$ in plane with a different constant)

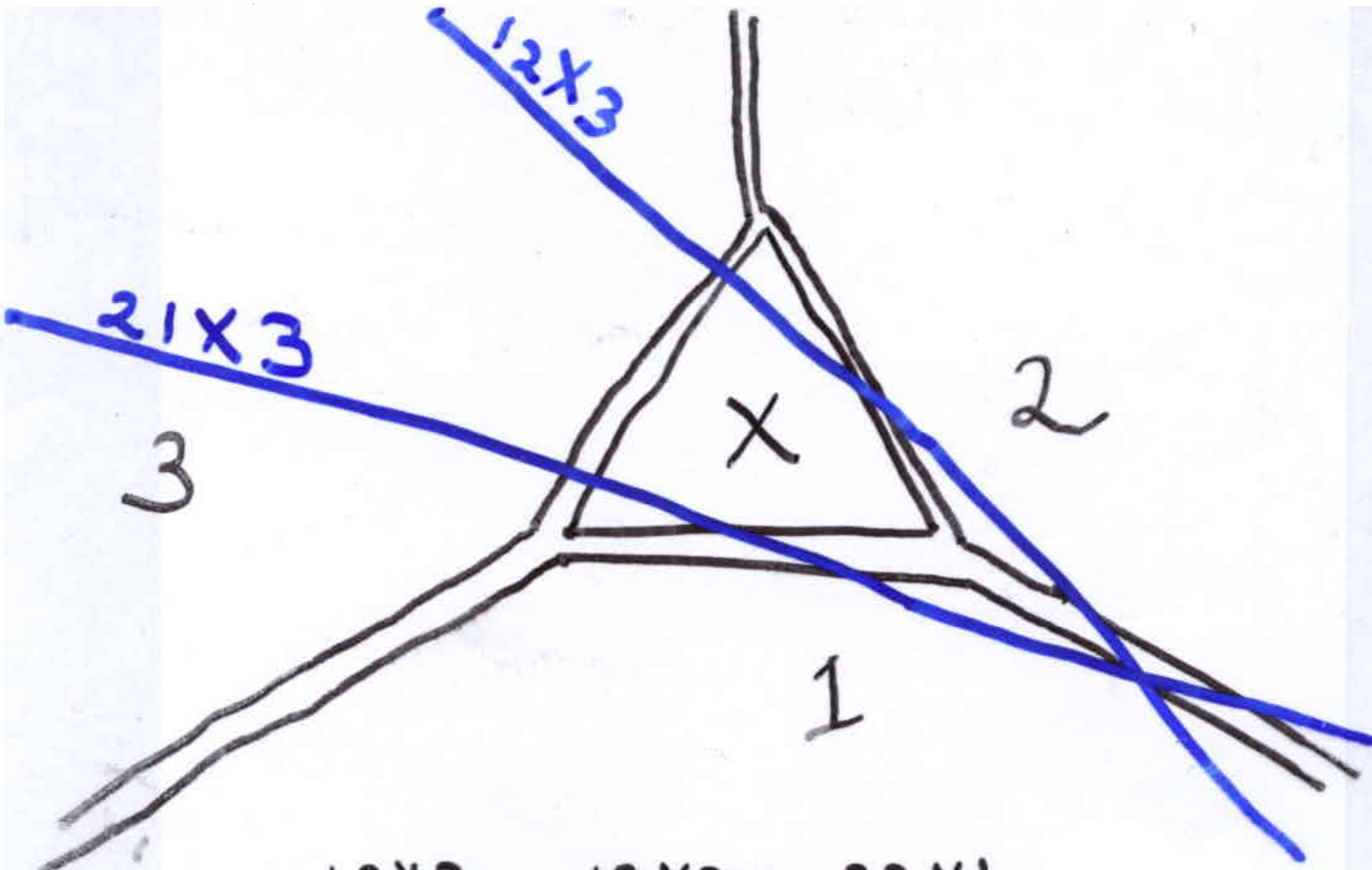
Ex 2. At most 4 geometric permutations for disjoint unit balls in $\mathbb{R}^d$ (Assuming there are many) (Suri + Zhou + K)

$+$ Convexity of certain cones

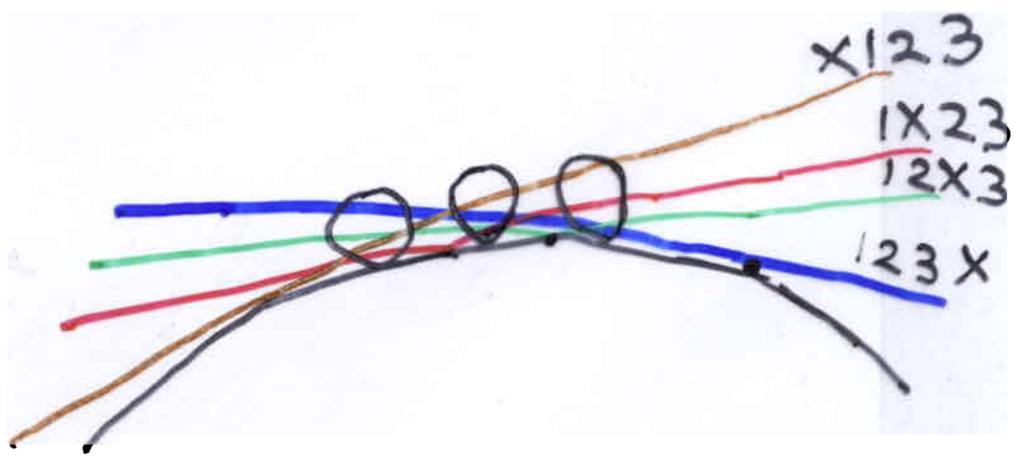
$\Rightarrow$ For disjoint unit balls in space;

If any k_0 admit a transversal then there is a transversal for all of them (Holmsen + Lewis + K).

More geometric permutations:



12X3 13X2 32X1
21X3 31X2 23X1



How many geometric permutations $\overbrace{\text{for}}^{\text{at most}}$ n-disjoint convex sets in R^d?

$O(n^{2d-2})$ (Wenger)

$\Omega(n^{d-1})$ (Lewis + K)

In plane: at most $2n-2$ (Edelsbrunner, Sharir)

at least $2n-2$ (Lewis, Zaks, K)

For translates in plane 3 (Lewis, Liu, K)

For balls in R^d: $\Theta(n^{d-1})$ (Smorodinsky, Sharir, Mitchell)

Fat convex 'fat' sets in R^d: $\Theta(n^{d-1})$ (Katz, Varadarajan)

Disjoint unit discs in R^2: at most 2 if n large (Sharir, Smorodinsky)

at most 2 if $n \geq 4$ (Holmsen, Asinowski, K)

At most 4 for unit balls in R^d if n large. (Suri, Zhou, K)

$1 \ldots k\, x\, y\, m \ldots n$

$1 \ldots k\, y\, x\, m \ldots n$

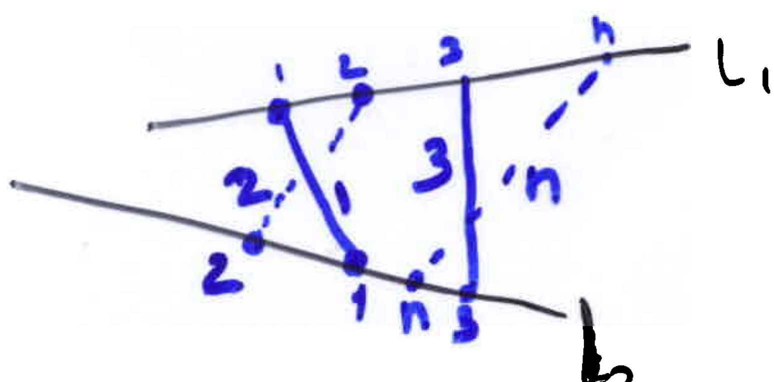
Forbidden families of geometric permutations in R^d

A family of (geometric) permutations is forbidden or non-realizable in R^d if there is no disjoint family of convex sets in R^d which has these permutations as geometric permutations.

Theorem 1: Each family of k permutations is realizable in R^{2k-1}

Follows from dimension arguements: by segments

Any 2 permutations are realizable in R^3



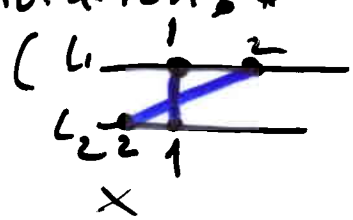
L_1 and L_2 are skew-lines

Theorem 2. For each k, there is a family of k permutations not realizable in R^{2k-2}

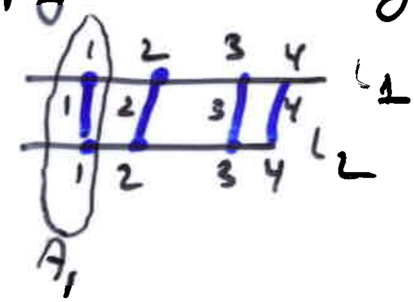
Examples:

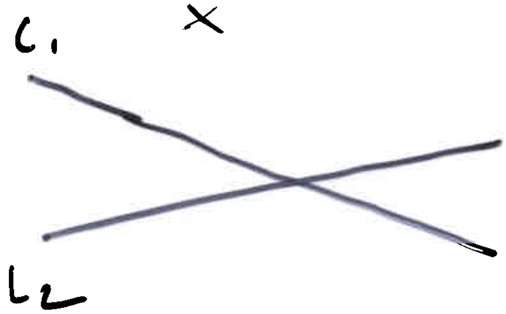
Example 1: 1234 and 2143 not realizable in plan

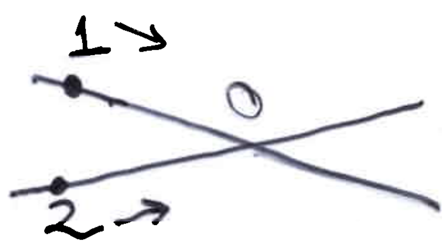
Parallel transversals imply the same geometric permutation, $:



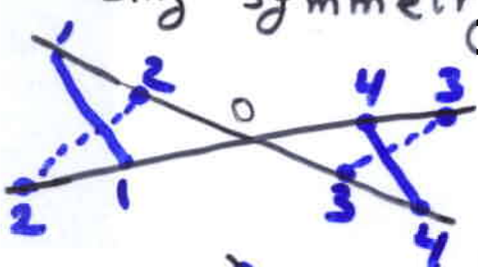
So





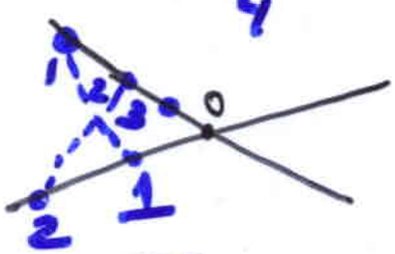


Using symmetry:



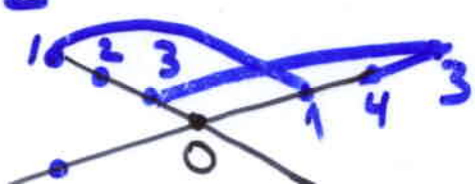
$A_1 \cap A_2 \neq \phi$, $A_3 \cap A_4 \neq \phi$

So σ



$A_1 \cap A_2 \neq \phi$

or



and $A_1 \cap A_3 \neq \phi$

Remark: Two permutations of $1, \dots, n$ are realizable as a pair of geometric permutations in the plane if they do not contain the pair

$$(i\,j\,k\,l) \quad (j\,i\,l\,k)$$

as subpermutations

Example 2. The triple

$l_1 : (123\ 456)$

$l_2 : (321\ 654)$

$l_3 : (246\ 135)$

is forbidden in R^3.

Example 3. The triple

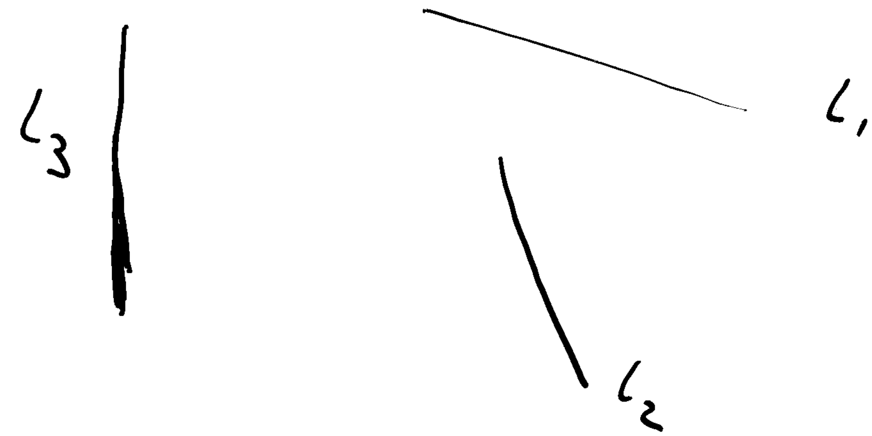
$(123 \quad 456 \quad 789)$

$(312 \quad 564 \quad 978)$

$(231 \quad 645 \quad 897)$

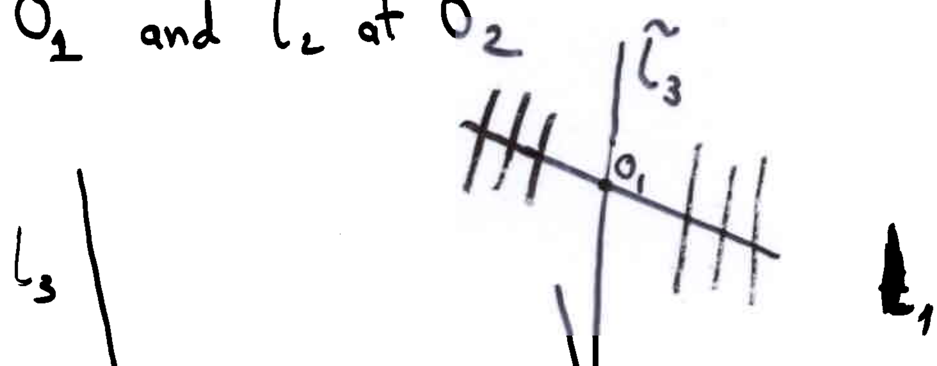
is forbidden in R^4.

Example 2:

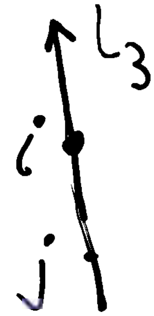


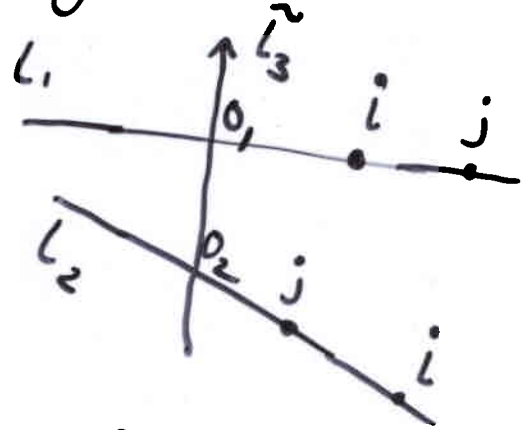
There is a translate $\tilde{l}_3$ of l_3 that meets

l_1 at O_1 and l_2 at O_2



l_3 is not necessarily a transversal.

However: If




Then, assuming l_3 and $\tilde{l}_3$ go up,

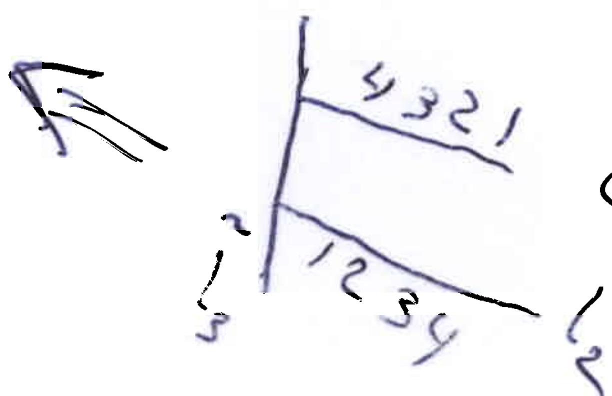
a plane separating A_i and A_j has A_i above A_j, on the same side as O_1 and O_2 on the same side as A_j.

So on l_3: i is above j.

(*)

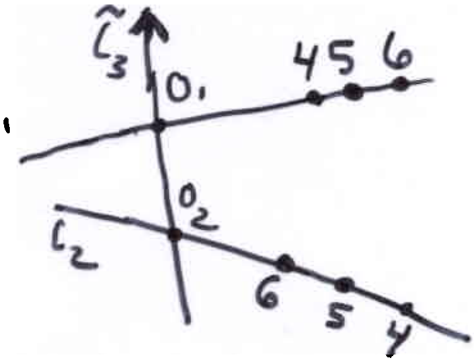
So if $O_1 \times i \times j$ on l_1
and $O_2 \times j \times i$ on l_2
then order of i and j on l_3 is as in l_1

(l_1 dominates l_2 for l_3)



Various cases to check, determind by relative position of O_1 and O_2 on l_1 and l_2 respectively:

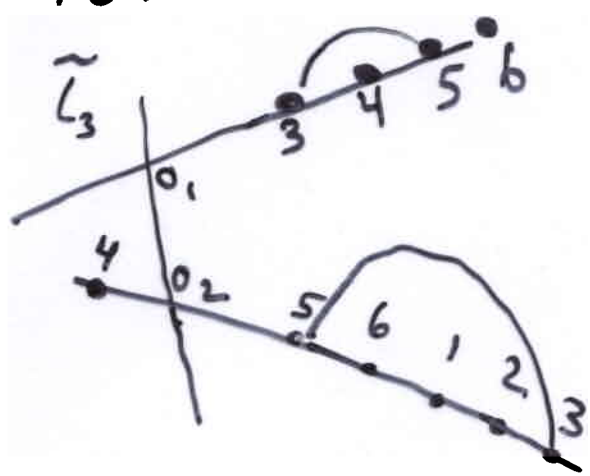
If $l_3 \uparrow$



$\Leftarrow$

in l_3 $4>5$, $5>6$ (or $4<5$, $5<6$)
or $4*5*6$
but in l_3 4 6 5 a contradiction

• If



Then in l_3:

$3>5$, $3>6$ (or $3<5$, $3<6$)

contradicting 6 3 5 in l_3.

⋮

All other cases lead to a contradiction

Example 3: A contradiction using:

There is a line in R^4 meeting both l_1 and l_2 and l_3. ...

Remarks: In R^3;

① For 3 permutations if choosing any one of them as l_3 ~~and~~ then (~~*~~) is satisfied Then 2 of the permutations coincide

② It is possible that these restrictions on triples of geometric permutations can lead to a proof of a constant bound on the number of geometric permutations for disjoint translates of a convex set in R^d

and to improved bounds on the number of geometric permutations for convex sets in R^d. (?)