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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu **Email/Phone:** aseceleanu2@math.unl.edu

Speaker's Name: Irena Swanson

Talk Title: Minimal Components over Certain Binomial Ideals

Date: 08/23/12 **Time:** 4:30 am / pm (circle one)

List 6-12 key words for the talk: binomial ideal, minimal primes algebraic statistics

Please summarize the lecture in 5 or fewer sentances: This lecture presents an analysis of minimal primes arising from certain statistical models and generalizations.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Minimal components over certain binomial ideals

joint work with Amelia Taylor
Julia Porcino (background with Reinhard Laubenbacher)
Alessio Sammartano

The main theme through my talk are structure theorems of minimal primes over various binomial ideals. The first set of ideals arose in algebraic statistics, and others are modifications of those: permanental ideals, and lastly determinantal ideals of generic s-~~Hankel~~ catalecticant matrices (a new concept introduced in Sammartano's work).

I. Algebraic statistics

... Alex Fink posted on the arXiv in 2009:

Discrete random variables $X_1, \dots, X_n$

X_i takes on a finite number of values

for simplicity the values are in the set $[r_i] = \{1, 2, \dots, r_i\}$

Def: X_1 and X_2 are independent if $\forall i \in [r_1]$ and $\forall j \in [r_2]$,

$$P(X_1 = i, X_2 = j) = P(X_1 = i) \cdot P(X_2 = j).$$

$\boxed{\text{Write } X_1 \perp\!\!\!\perp X_2}$ Thus

$$\underbrace{\left[P(X_1 = i, X_2 = j)\right]_{ij}}_{r_1 \times r_2 \text{ matrix}} = \begin{bmatrix} P(X_1) = 1 \\ P(X_1) = 2 \\ \vdots \\ P(X_1) = r_1 \end{bmatrix} \begin{bmatrix} P(X_2 = 1) & \cdots & P(X_2 = r_2) \end{bmatrix}$$

has rank ≤ 1, and - assuming nonzero, has rank $= 1$.

So $I_2(P(X_1 = i, X_2 = j)) = 0$.

Conversely, $I_2(P(X_1 = i, X_2 = j)) = 0 \implies X_1$ is indep. of X_2.

X_1 and X_2 are <u>conditionally independent given</u> X_3 $(X_1 \perp\!\!\!\perp X_2 | X_3)$
if $\forall k \in [r_3]$, $P(X_1 = i, X_2 = j, X_3 = k)_{ij}$ = ~~P(X~~ has rank 1.

Notation board: $N = [r_1] \times [r_2] \times \cdots \times [r_n]$
$R = k[X_a : a \in N]$ a poly ring in $r_1 r_2 \cdots r_n$ vars over field
$M =$ ~~k~~ $[X_a]_{a \in N}$ $\leftarrow$ ~~an~~ generic n-dim'l hypermatrix

If $n = 2$: $X_1 \perp\!\!\!\perp X_2 \iff I_2(M) = 0$
allow only non-neg real ~~zeros~~ sols to $\leftarrow$ variety over arb. field k

Moral: algebraic statistics uses methods of algebra/geometry for some statistics models, but the algebraic solutions may need some adoptotition.

Moral II: not all ~~the algel~~ statistics problems can be modeled by algebra.

Moral III: not all the algebra needed for statistics has been developed, so statistics is useful to algebra for helping advance algebra.

If $n = 3$: $X_1 \perp\!\!\!\perp X_2 | X_3 \iff \sum_{k=1}^{r_3} I_2(k^{th}$ ~~floor~~ floor/level of $M) = 0$

These are binomial prime ideals. Well-known (det. variety, sums of prime ideals in disjoint variables over an alg. closed field).

If $n = 3$: $X_1 \perp\!\!\!\perp X_2 \iff I_2(M') = 0$, where $(M')_{ij} = \sum_k M_{ijk}$.
(not binomial directly, but is binomial after a change of variables.

Why do I worry about binomiality: Eisenbud–Sturmfels: better understood structure (theory), and in practice: Thomas Kahle binomial package in M2. All ideals in the sequel are binomial. I will say no more about binomiality.

Intersection axiom from statistics: ~~$X_1 \perp\!\!\!\perp X_2 \mid X_3$ and X~~

$X_1 \perp\!\!\!\perp X_2 \mid X_3$ and $X_1 \perp\!\!\!\perp X_3 \mid X_2$ $\Longleftrightarrow$ $X_1 \perp\!\!\!\perp \{X_2, X_3\}$

axiom, but only in the interior of the probability simplex, i.e. when all $p(X_1=i, X_2=j, X_3=k) > 0$.

$\sum_{k=1}^{r_3} I_2(k^{th}$ floor of $M)$ $\qquad$ $\sum_{j=1}^{r_2} I_2(j^{th}$ wall of $M)$

$I_2(r_1 \times r_2 r_3$ matrix, $) = 0$
rows indexed by $[r_1]$
and columns by $[r_2 r_3]$
"flattening of M"

~~Cartwright and~~
~~Engström and Cox~~ ~~conjectured~~

Algebraically, the axiom says

$$\sqrt{\sum_{k=1}^{r_3} I_2(k^{th}\text{ floor of } M) + \sum_{j=1}^{r_2} I_2(j^{th}\text{ wall of } M)} = \sqrt{I_2(r_1 \times r_2 r_3)}$$

after inverting ~~the~~ the product of the variables.

Statistics: Hammersley-Clifford know more: remove radical sign and equality still holds.
Cartwright and Engström: conjectured what the other minimal ~~com~~ components primes over $\sum_{k=1}^{r_3} + \sum_{j=1}^{r_2}$ are.
Fink proved their conjecture.

Swanson-Taylor, Ay-Rauh generalized it, in different ways. And there is room for many other generalizations.

II. Notation for algebra: $a, b \in \mathbb{N}$, $i \in [n]$, "switch" $s(i,a,b) \in \mathbb{N}$:

$$s(i,a,b)_j = \begin{cases} a_j & \text{if } j \neq i \\ b_i & \text{if } i = j. \end{cases}$$

$f_{iab} = x_a x_b - x_{s(i,a,b)} x_{s(i,b,a)}$

$d(a,b) =$ Hamming distance from a to b $= \#\{i : a_i \neq b_i\}$

$t \in [n]$ $I^{\langle t \rangle} = (f_{iab} : i \leq t, d(a,b) = 2)$

$\tilde{I}^{\langle t \rangle} = (f_{iab} : i \leq t)$.

Intersection axiom above: $I^{\langle 1 \rangle} : (\prod_a x_a)^\infty = \tilde{I}^{\langle 1 \rangle}$.

Results: Fink: $n=3$, $t=1$

S+Taylor: arb n, $t \le n$

~~Sammo~~

S+Porcino: arb n, $t \le n$; but 2×2 permanental ideal

Ay-Rauh: arb n, $t=1$

Sammartano: arb n, t, for generic S-catalecticant hypermatrices

[Technicalities follow — try to make it light]

Definition: $S \subseteq N$ is t-switchable if $\forall a,b \in N$ $\forall i \le t$,

$d(a,b) = 2 \Rightarrow s(i,a,b) \in S$.

Note: $(n-1)$-switchable $=$ n-switchable.

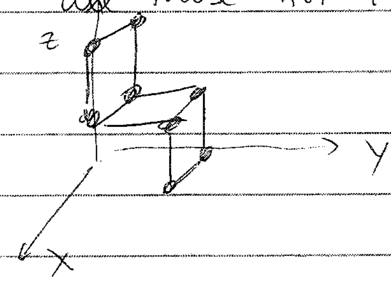
Examples:

$n=2$: $S = S_1 \times S_2$ (only rectangles)

$n=3$, $t=2$: S is a disjoint union of boxes, ~~that do n~~ where the boxes have no parts of coordinates in common:

Say $\{(1,1,1)\} \cup \{(2,2,2)\}$

$t=1$: all those for $t=2$ and more



← this is not 2-switchable but it is 1-switchable

Definition: $a,b \in S$ are connected in S if $\exists a_0=a, a_1, \dots, a_m=b \in$ such that $\forall i=1,\dots,a_m$, $d(a_{i-1},a_i)=1$. This is an equivalence relation.

Define for S t-switchable:

$$Var_S = (x_a : a \notin S)$$

$$\tilde{I}_S^{\langle t\rangle} = (f_{iab} : a,b \in S \text{ t-conn.}, i \le t)$$

$$P_S^{\langle t\rangle} = Var_S + \tilde{I}_S^{\langle t\rangle}.$$

EXPAND

Fink ST

Main result:

radical if $t=1$; ← Ay-Rauh, Fink otherwise usually not radical.

$$\sqrt{I^{\langle t\rangle}} = \bigcap_{S \text{ t-switchable}} P_S^{\langle t\rangle}$$

← each is a prime ideal, we have a GB under any diag. or ... Com ... redundant ones

Permanental results:

$$\sqrt{\text{permanental } I^{\langle t \rangle}} = \bigcap \left(\mathrm{Var}_S + (X_a X_b + (\pm 1)\, X_{S(i,a,b)} X_{S(\ldots\right.$$

$a_i \neq b_i$ $\quad d(a,b) \geq 2 \quad$ $a, b \in S$ con... $\quad i \leq t$...

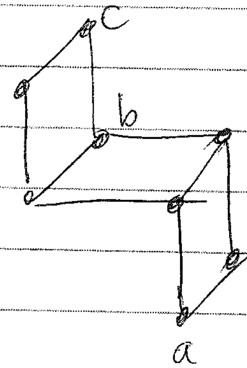
$(-1)^{\text{path length from } a \text{ to } b, \ldots}$

~~those S-switchable~~ in which path lengths have well-defined parity.

again redundancies

Proofs relatively similar, the hardest part is keeping track of signs.

Back to



This S does give a min. prime ideal in $[2] \times [2] \times [3]$ for $t =$

$f_{1,a,b}$

$X_a X_c + X_{S(1,a,c)} X_{S(1,c,a)}$ } are the corr. prime

Special case: $n = 2$ (cf. Laubenbacher ~1998)

for simplicity $r_1, r_2 \geq 3$

The only 1-permanental switchable subsets of $[r_1] \times [r_2]$ are

2×2 submatrix

redundant part of a row $\longleftarrow$ full row

part of a column $\longleftarrow$ full column

So this gives a new perspective on ~~the~~ some old work.

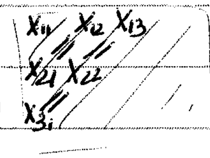
Sammartano:

Further structure on M: s-catalecticant:

$$x_a = x_b \iff \sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i, \ a_{s+1} = b_{s+1}, \dots, a_n = b_n$$

If $n=2$: 1-cat = generic

2-cat = catalecticant (in the classical sense)



If $n=3$: waive hands.

$$\sqrt{I^{\langle s,t\rangle}} = \bigcap_{S\ (s,t)\text{-switchable}} P_S^{\langle t\rangle}$$

& omit redundancies

You can guess the definition

But to prove that these are the min primes takes nontrivial modifications.

Sammartano also gives more detailed info on the primary decomp. when $s=n$ and $s=n-1$.

Exercises

1. Check that if $B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ & we denote the initial cluster variables as (y_1, y_2), then the set of all cluster variables is in bijection with $\mathbb{Z}$ (denote them by y_m, $m \in \mathbb{Z}$), & they satisfy:

$$y_{m-1} y_{m+1} = \begin{cases} y_m^b + 1 & m \text{ odd} \\ y_m^c + 1 & m \text{ even} \end{cases}$$

2. For which b and $c \in \mathbb{Z}$ is the set of cluster variables finite?

3. Prove the following claim.
<u>Claim:</u> Let T be a triangulation, and T' be new triangulation obtained by flipping t_i. Then
$$B(T') = \mu_i(T).$$

4. Explicitly compute all cluster variables & seeds associated to the cluster algebra coming from a ~~pentagon hexagon~~ pentagon.
(Challenge: hexagon instead.)