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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu   Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Irena Peeva

Talk Title: Infinite free resolutions

Date: 08/30/12   Time: 10:30 am / pm (circle one)

List 6-12 key words for the talk: Betti numbers, resolution, Koszul, complete intersection

Please summarize the lecture in 5 or fewer sentances: Much less is known about properties of infinite free resolutions than those of finite free resolution. The lectures discuss resolutions over complete intersections and Koszul rings.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Irena Peeva – Infinite free resolutions III

References: Avramov – Problems on infinite free resolutions (1992)
Conca, DeNegri, Rossi – Koszul rings and regularity (2012)

Koszul rings and complete intersections

Koszul rings were introduced by Priddy (1970)

Def R is Koszul if $\mathrm{reg}_R(k) = 0$ (i.e. k has a linear minimal free resolution).

Prop If $\frac{I}{\subseteq m^2}$ is Koszul then I is generated by quadrics

Roos (1993) $S = k[x,y,z,u,v,w]$

Fix a gen. set $I = (x^2, xy, yz, z^2, zu, u^2, uv, vw, w^2,$
$xz + qzw - uw,\ zw + xu + (q-2)uw)$

$\Rightarrow b_{i,j}^R(k) = 0$ for $i \neq j$ and $i \leq q$

$b^R_{q+1,q+2}(k) \neq 0$

1) If R is Koszul, then the min. free res. of k is the generalized Koszul complex.

2) $P_k^R(t) = \frac{1}{Hilb(-t)}$, in particular $P_k^R(t)$ is rational

$\Longleftrightarrow P_k^R(t)\ Hilb(-t) = 1$

$\sum b_i(M) t^i = \sum_i (\dim Ext^i(k,R)_i) t^i = Hilb_{\sum Ext^i(k,k)_i}(t)$

3) Froberg (1975)
If I is generated by quadratic monomials $\Rightarrow$ I is Koszul

4) Veronese rings and Segre rings are Koszul
$V_{s,r} = k[$all monomials of degree r in s variables$]$
[Bărcănescu – Manolache 1981]

Tool 1: if I has a quadratic Gröbner basis $\Rightarrow$ I is Koszul

Tool 2: Koszul filtrations (Conca – Trung – Valla 2001)

5) <u>The pinched Veronese</u>
$V = k$ [all mon. of degree 3 in x, y, z except xyz]
Caviglia (2009) V is Koszul
Conca - Caviglia (2012) - more general result

$\mathcal{A}$ = central hyperplane arrangement $= \bigcup_{i=1}^{m} H_i$
Study the topology of the complement $X = \mathbb{C}^m \setminus \mathcal{A}$
$H^*(X) \cong$ Orlik - Solomon algebra
E = exterior algebra in $e_1, \dots, e_m$
$J = \left(\partial(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{q=1}^{p} (-1)^{q-1} e_{i_1} \wedge \dots \wedge \hat{e}_{i_q} \wedge \dots \wedge e_{i_p} \right)$
for $(i_1, \dots, i_p)$ such that codim $(H_{i_1} \cap \dots \cap H_{i_p}) < p$

Lower Central Series (LCS) formula holds when A is Koszul
<u>Problem</u> Find a central hyperplane arrangement for which $H^*(X)$ is Koszul but there is no quadratic Gröbner basis.

<u>Complete Intersections</u>
Assume I is generated by a homogeneous regular sequence.
1. Gulliksen (1974) $\forall M$, $P_M^R(t)$ is rational.
2. Eisenbud (1980) If the Betti numbers of M are bounded $\Rightarrow$ the min free res. of M is eventually periodic of period 2 and the Betti numbers are constant.
3. Avramov - Gasharov - Peeva (1998)
 - $\forall M$, $\{b_i^R(M)\}$ is eventually non-decreasing
 - $\exists$ 2 polynomials f, g s.t. $b_{2i}^R(M) = f(i)$, $b_{2i+1}^R(M) = g(i)$ for $i \gg 0$, $\deg f = \deg g$ and f and g have the same leading coefficient.

1) We have an "odd" and "even" patterns
2) We study the asymptotic behavior
Eisenbud (1980) constructed examples with complicated behavior at the beginning of the resolution

Tuct: Eisenbud operators

$\mathrm{Ext}_R(M,k)$ is finitely generated over $k[\chi_1,\dots,\chi_c]$
where χ_i = Eisenbud operators
c = codim

Structural results:

1. The min free res of k (Tate 1957)
$\mathcal{D}(y_1 \dots y_c) \otimes \mathbb{K}$
divided powers algebra Koszul complex
$f_j = a_{j1} x_1 + \dots + a_{jn} x_n$

$$R^{c} \longrightarrow R^n \xrightarrow{(x_1,\dots,x_n)} R$$
$$e_1, \dots, e_n \longmapsto x_i$$
$$y_j \longmapsto a_{j1} e_1 + \dots + a_{jn} e_n \longmapsto f_j = 0$$

2. Matrix factorization
Over a hypersurface S/f, $\forall M$ $\exists \alpha, \beta : S^p \to S^p$ s.t. $\alpha\beta = \beta\alpha = f\,\mathrm{id}$ that asymptotically give the min free resol. of M over $R = S/(f)$.

$$\dots\dots \xrightarrow{\bar{\alpha}} R^p \xrightarrow{\bar{\beta}} R^p \xrightarrow{\bar{\alpha}} R^p$$