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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecelearu2@math.unl.edu

Speaker's Name: Karen Smith

Talk Title: Introduction to Frobenius Splitting #3

Date: 08/30/12 Time: 2:00 am / pm (circle one)

List 6-12 key words for the talk: F-regularity, test ideal, symbolic powers

Please summarize the lecture in 5 or fewer sentances: This lecture discusses test ideals as obstructions to F-regularity. An application to symbolic powers is given.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Karen Smith — Frobenius splitting III

Test ideals $I \subseteq k[x_1, \dots, x_n]$ say I = ideal of functions vanishing on $Z \subseteq \mathbb{A}^d_n$

Def $I^{(N)} \subseteq k[x_1, \dots, x_d]$ ideal of functions vanishing to order N at every point of Z.

Obviously $I^N \subseteq I^{(N)} = P_1 \cap \dots \cap P_n$.

$\underbrace{(P_1 \cap \dots \cap P_n)}_{\text{minimal primary components of } I^N} \cap \underbrace{(Q_1 \cap \dots \cap Q_k)}_{\text{embedded primary components}}$

Thm (Swanson)

$\exists K \in \mathbb{N}$ (depending on I) s.t. $I^{(KN)} \subseteq I^N$, $\forall N$.

Question: Is there a uniform K that works for $I \subseteq k[x_1, \dots, x_d]$?

Answer: Yes!

Thm $I \subseteq k[x_1, \dots, x_d]$ unmixed ideal. Then $I^{(dN)} \subseteq I^N$, $\forall N$.

- char 0: Ein, Lazarsfeld, Smith — using multiplier ideals
- char p: Hochster — Huneke

Today: nice proof of this Thm in char p, using test ideals.

Generalities on test ideals (Hara — Yoshida)

R char $p > 0$ ($R \hookrightarrow R^{1/p}$ finite ext.)

$\mathfrak{a} \subseteq R$ $\forall t \in \mathbb{Q}_{>0} \rightsquigarrow \{\tau(R, \mathfrak{a}^t)\}_{t \in \mathbb{Q}_{>0}}$

- "classical" commutative algebra case $\mathfrak{a} = (1)$, $\tau(R) =$ the Hochster - Huneke test ideal
- classical algebraic geometry $(R, \mathfrak{a}^t) \rightsquigarrow \tau(\mathfrak{a}^t)$, R regular ring

Properties of $\{\tau(R, \mathfrak{a}^t)\}_{t \in \mathbb{Q}_{>0}}$

1) $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \tau(R, \mathfrak{a}^t) \subseteq \tau(R, \mathfrak{b}^t)$
2) $t > t' \Rightarrow \tau(R, \mathfrak{a}^{t'}) \supseteq \tau(R, \mathfrak{a}^t)$

If R is regular (or even F-regular)

3) $\mathfrak{a} \subseteq \tau(\mathfrak{a})$

4) (Skoda) $\mathfrak{a} = (g_1, \dots, g_r) \Rightarrow \tau(\mathfrak{a}^r) = \mathfrak{a}\,\tau(\mathfrak{a}^{r-1})$

$\tau(\mathfrak{a}^{r+N}) = \mathfrak{a}\,\tau(\mathfrak{a}^{r+N-1})$ $N \in \mathbb{Q}_{\geq 0}$

5) "Restriction thm" $x \in R$ $x \neq 0$

$\bar{\mathfrak{a}}$ denotes image of $\mathfrak{a}$ mod x in $R/(x)$

$\tau(\bar{\mathfrak{a}}^t) \subseteq \tau(\mathfrak{a}^t)\, R/(x)$

6) "Subadditivity Thm" $\mathfrak{a}, \mathfrak{b}$ $\tau(\mathfrak{a}^t \mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t)\,\tau(\mathfrak{b}^s)$

<u>Def</u> (Blickle, Mustaţă, Smith)

R regular

R^{1/p^e} as R-module is locally free; localize and assume R^{1/p^e} free

First consider $t = \frac{n}{p^e}$, $\mathfrak{a}^t = \mathfrak{a}^{n/p^e}$ in R^{1/p^e}

View $\mathfrak{a}^t R^{1/p^e}$ as R-module

<u>Def</u> (for $t = \frac{n}{p^e}$) $\tau(R, \mathfrak{a}^{\frac{n}{p^e}})$ = images of $\mathfrak{a}^t R^{1/p^e}$ under all $\varphi \in \mathrm{Hom}(R^{1/p^e}, R)$

$$\tau(R, \mathfrak{a}^{\frac{n}{p^e}}) = \langle \varphi(g^{\frac{1}{p^e}}),\ g \in \mathfrak{a}^n,\ \varphi \in \mathrm{Hom}_R(R^{\frac{1}{p^e}}, R) \rangle$$

<u>Rk</u> 1) This is independent of how we write t as a fraction.

2) If $t > t'$ both with p^e denominator $\tau(\mathfrak{a}^t) \subseteq \tau(\mathfrak{a}^{t'})$

Now: given arbitrary $t \in \mathbb{Q}_{>0}$, choose a decreasing sequence $\{t_i\}$ of rational numbers with denominator a power of p converging to t.

$\tau(\mathfrak{a}^{t_1}) \subseteq \tau(\mathfrak{a}^{t_2}) \subseteq \dots \subseteq \dots$ must stabilize to $\tau(\mathfrak{a}^t)$ (Noeth. Property)

3) $\tau(\mathfrak{a}^t)$ is independent of the choice of converging sequence

<u>Def</u>: $\tau(R, \mathfrak{a}^t) = \bigcup_e \tau\left(R, \mathfrak{a}^{\frac{\lceil p^e t \rceil}{p^e}}\right)$

Hint for Property 4: $\mathfrak{a} = (g_1, \dots, g_r)$, $\mathfrak{a}^{rp^e - r + 1} \subseteq (g_1^{p^e}, \dots, g_r^{p^e})\,\mathfrak{a}^{rp^e - r - p^e + 1}$

Hint for Property 5.

$$\begin{tikzcd} R^{\frac{1}{2}} \arrow[r, dashed, "\tilde{\varphi}"] \arrow[d] & R \arrow[d] \\ (R/(x))^{1/p^e} \arrow[r, "\varphi"] & R/(x) \end{tikzcd}$$

Hint for Property 6: $\mathfrak{a} \subseteq k[x_1, \dots, x_d]$, $\mathfrak{b} \subseteq k[y_1, \dots, y_{d'}]$

$\tau(a^s b^t) = \tau(a^s)\,\tau(b^t)$

Mod out $x_1 - y_1, x_2 - y_2, \ldots, x_i - y_i$ and use Property 5

<u>Proof of Thm</u> (sketch)

"Asymptotic test ideal"

$\{I_n\}_{n \in \mathbb{N}}$, $I_n I_m \subseteq I_{m+n}$ $\forall m, n \in \mathbb{N}$

Symbolic powers satisfy this property $I_n = I^{(n)}$.

Observe $\tau(R, I_n) \subseteq \tau(R, I_{mn}{}^{1/m})$

<u>Def</u> $\tau_\infty(I_n) = \bigcup_{m \in \mathbb{N}} \tau(R, I_{mn}{}^{1/m}) = \tau(R, I_{mn}{}^{1/m})$ for m sufficiently divisible

Consider $\{I^{(N)}\}_{N \in \mathbb{N}}$.

$$\underbrace{I^{(Nd)} \subseteq}_{\text{ex using Property 3}} \tau_\infty(I^{(Nd)}) \underbrace{\subseteq}_{\text{ex using Property 6}} [\tau_\infty(I^{(d)})]^N \underbrace{\subseteq}_{\text{show } \tau_\infty(I^{(d)}) \subseteq I \text{ by Property 4}} I^N$$

<u>Classical commutative algebra setting</u> $a = (1)$

Define $\tau(R)$. This will be $\tau(R) = R \Leftrightarrow R$ is F-regular

R char $p > 0$ (R not regular)

<u>Def</u> $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. Say $J \subseteq R$ is φ-compatible if $\varphi(J^{1/p^e}) \subseteq J$.

$$\begin{tikzcd} R^{1/p^e} \arrow[r, "\varphi"] \arrow[d] & R \arrow[d] \\ (R/J)^{1/p^e} \arrow[r, dashed] & R/J \end{tikzcd}$$

One interesting case: φ is a splitting of $R \hookrightarrow R^{1/p^e}$.

Then a φ-compatible ideal is a <u>compatibly split</u> ideal.

<u>Def</u>: The test ideal $\tau \subseteq R$ is the smallest $\underset{\text{non-zero}}{\varphi\text{-compatible}}$ ideal of J for <u>all</u> $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ s.t. $\varphi(J^{1/p^e}) \subseteq J$ $\forall\, \varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$.

<u>Thm</u> R is F-reg $\Longleftrightarrow \tau(R) = R$.