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# NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean   Email/Phone: asecelean2@math.uiuc.edu

Speaker's Name: Bernard Leclerc

Talk Title: Preprojective algebras and Lie theory

Date: 08/30/12   Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: preprojective algebra, Lie theory

Please summarize the lecture in 5 or fewer sentences: Several cluster algebras appear in Lie theory (e.g. Grassmanians). They can be understood by relating them to certain categories of modules over a preprojective algebra.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - Computer Presentations: Obtain a copy of their presentation
    - Overhead: Obtain a copy or use the originals and scan them
    - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Bernard Leclerc - Preprojective algebras and Lie theory III

~~[illegible]~~

Def $X \in \mathrm{mod}\,\Lambda$ is rigid if $\mathrm{Ext}^1_\Lambda(X,X)=0$

Thm (Geiss - Schröer)

Let $r = \dim N$.

The number of pairwise non isomorphic indecomposable summands of a rigid $\Lambda$-module is $\leq r$.

Def $X$ is a maximal rigid module if it has $r$ non-isomorphic indecomposable summands

Ex $A_2$ $r=3$ two basic maximal rigid modules:

$T = S_1 \oplus P_1 \oplus P_2$, $T' = S_2 \oplus P_1 \oplus P_2$

Fix $\underline{i}$ a reduced decomposition of $w_0 \in W$:

$w_0 = s_{i_1} s_{i_2} \dots s_{i_r}$ $\rightsquigarrow$ BFZ - initial seed for $\mathbb{C}[N]$

$\Sigma_{\underline{i}} = ((f_1^{(\underline{i})}, \dots, f_r^{(\underline{i})}); Q_{\underline{i}})$

Thm [Geiss - Leclerc - Schröer]

There exists a maximal basic rigid module

$V_{\underline{i}} = V_1^{(\underline{i})} \oplus \dots \oplus V_r^{(\underline{i})}$ s.t. $\varphi_{V_k^{(\underline{i})}} = f_k^{(\underline{i})}$, $k=1,\dots,r$

Moreover, let $Q_{V_{\underline{i}}}$ be the Gabriel quiver of $\mathrm{End}_\Lambda(V_{\underline{i}})^{V_{\underline{i}}}$, then $Q_{V_{\underline{i}}} = Q_{\underline{i}}$. (up to arrows connecting vertices corresponding to projective summands of $V_{\underline{i}}$.)

Let $\mathcal{E}_{\underline{i}}$ be the stable endomorphism algebra of $V_{\underline{i}}$.

By a Thm of Buan - Iyama - Reiten - Smith, $\mathcal{E}_{\underline{i}}$ is given by a quiver with potential.

$\mathcal{F}_{X,\underline{i}^a} \simeq Gr_{\underline{e}}$ (representation of $\mathcal{E}_{\underline{i}}$)

## Mutations of maximal rigid modules

**Thm** [GLS] Let $T = T_1 \oplus \dots \oplus T_r$ be basic maximal rigid.
Let $T_k$ be a non-projective summand.
$\exists!\ T_k^* \not\cong T_k$ s.t. $T^* = (T/T_k) \oplus T_k^*$ is again maximal rigid.

$\Sigma_T = (\varphi_{T_1}, \dots, \varphi_{T_r}, Q_T)$
$\downarrow \mu_k$ = Fomin - Zelevinsky cluster mutation
$\Sigma_{T^*} = ( \text{---}'' \text{------}, Q_{T^*})$

$\left\{ \begin{array}{l} \text{maximal rigid modules} \\ \text{(which can be reached from} \\ V \text{ by a sequence of mutations)} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{(seeds)} \\ \text{clusters} \\ \text{of} \\ \mathbb{C}[N] \end{array} \right\}$

$T \longmapsto \Sigma_T$

## Relation with canonical basis

**Thm** (Lusztig)
(i) $\Lambda_{\underline{d}}$ is an equidimensional variety
(ii) the number of irreducible components of $\Lambda_{\underline{d}}$ =
= $\dim(U(n)_{\underline{d}})$ where $U(n)$ has $\mathbb{N}^I$-grading by
$\deg e_k = (0, \dots, \underset{k^{th}\text{ position}}{1}, 0, \dots, 0)$
(iii) For every irreducible component $Z$ of $\Lambda_{\underline{d}}$, there is a unique $f_Z \in M_{\underline{d}}$ s.t. $f_Z(x) = \begin{cases} 1 & \text{generically on } Z \\ 0 & \text{generically on } Z' \neq Z \end{cases}$
$\rightsquigarrow \{ f_Z \mid Z = \text{irreducible component of } \Lambda_{\underline{d}}, \forall \underline{d} \}$

Let $Z$ irred. component of $\Lambda_{\underline{d}}$

$\rho_Z : \begin{pmatrix} M_d \longrightarrow \mathbb{C} \\ f \longmapsto \text{generic value of } f \text{ on } Z \end{pmatrix} \in M_d^*$

$\{\rho_Z\}$ dual semicanonical basis

Ex Suppose $T$ is a rigid module of dim vector $\underline{d}$.
$\Leftrightarrow$ the $GL_{\underline{d}}$-orbit of $T$ in $\Lambda_{\underline{d}}$ is open.
If $T \in Z$ irreducible component $\rho_Z = \rho_T$.
Hence for every rigid module $T$, $\varphi_T$ belongs to the dual semicanonical basis of $U(n)$.
$\Rightarrow$ every cluster monomial belongs to the dual semicanonical basis $\underbrace{\phantom{x}}_{\text{of } \mathbb{C}[N]}$

$\mathcal{S}$=semicanonical basis $\longleftarrow$ Construction of $U(n)$ as a convolution algebra of constructible functions on $\Lambda_{\underline{d}}$

$\mathcal{B}$=canonical basis $\longleftarrow$ Construction of $U_q(n)$ as a convolution of complexes of constructible sheaves on $rep(Q,\underline{d})$

Thm [GLS] $\mathcal{B}|_{q=1} = \mathcal{S}$ iff $G$ is of type $A_n$, $n \leq 4$.