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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean Email/Phone: asecelean2@math.uiuc.edu

Speaker's Name: Alek Vainshtein

Talk Title: Cluster algebras and Poisson geometry

Date: 09/04/12 Time: 3:30 am/pm (circle one)

List 6-12 key words for the talk: cluster algebra, Poisson geometry, Poisson bracket.

Please summarize the lecture in 5 or fewer sentances: This lecture relates cluster algebras to Poisson geometry.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Cluster Algebras and Poisson Geometry #2 9/4/12

Alek Vainshtein

① Grassmannian $Gr_{k,n}$ example
② Grassmannian revisited.

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①

Philosophy

We want to find cluster structure in the ring of regular functions on V ("Zariski open in $\mathbb{C}^{n+m}$) e.g. "type nice" assuming there is Poisson structure on V. variety.

Steps:

① Find a log-canonical basis (in $\mathcal{O}(V)$)
$\{x_1, \dots, x_{n+m}\}$

② Find the exchange matrix B.

We know $\Omega^{[1,n+m]}_{[1,n]} = \lambda B$

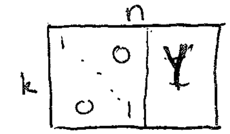
$B\,\Omega = \lambda [I\,O]$

③ check that all x_i belong to $\mathcal{O}(V)$.

Claim ①+②+③ $\Rightarrow \mathcal{A}(B)$ the cluster algebra
$\subseteq \mathcal{A}(B)$ the upper-cluster algebra
$\subseteq \mathcal{O}(V)$

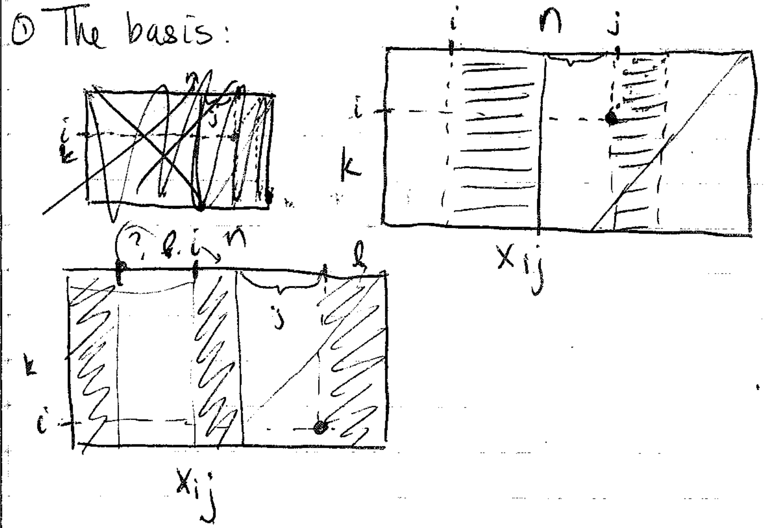
④ Prove that any regular function is in $\mathcal{A}(B)$
$\Rightarrow \mathcal{A}(B) = \mathcal{O}(V)$

Standard (Sklyanin) bracket



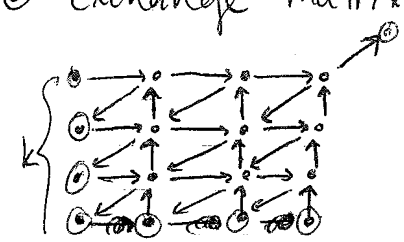
$$\{y_{ij}, y_{\alpha\beta}\} = \frac{1}{2}\left(\mathrm{sign}(\alpha - i) - \mathrm{sign}(\beta - j)\right) y_{i\beta}\, y_{\alpha j}$$

① The basis:



② Claim: $\{x_{ij} \mid 1 \le i \le k,\ 1 \le j \le m\}$ is ~~log~~ log-canonical basis.

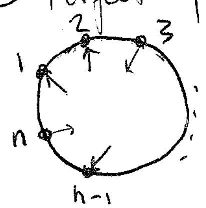
② Exchange matrix $x^{[1,k]}_{[1,k]}$



$k=4,\ m=4$

③, ④: Slide animation

(2) Perfect planar networks on a disc (Postnikov)

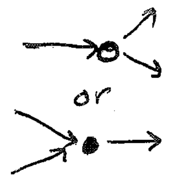


n boundary vertices

$I \subset [n]$ are sources

$J = I^c$ are sinks

All internal vertices are

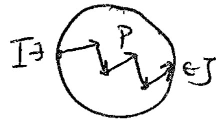


$\mathcal{N}_{I,J}$ = all networks with sources I, sinks J

Edge weights w_e

The space of edge weights of a given network N is denoted $\mathcal{E}_N = (\mathbb{R} \setminus 0)^{|E|}$

path weights: path P between boundary vertices



$$w_P = \pm \prod_{e \in P} w_e$$

If P has

we add some signs based on # repetitions.

boundary measurement:

$$b_{ij} = \sum_{P\, i \rightsquigarrow j} w_P$$

Claim (Postnikov) b_{ij} is a (subtraction free) rational function in edge weights.

boundary measurement matrix:

a $k \times (n-k)$ matrix, $k = |I|$, where b_{ij} is the boundary measurement.

boundary measurement map $\beta_N : \mathcal{E}_N \to Mat_{k,m}$

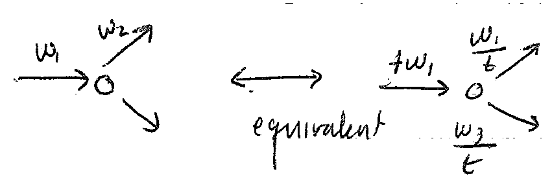
(1) $\{\ ,\ \} = 0$ if edges are not adjacent

(2) $\{\ ,\ \}$ depends only on color of node

(3) "good behavior" under concatenation. ↳ tutorial.

Call such $\{\ ,\ \}$ universal brackets

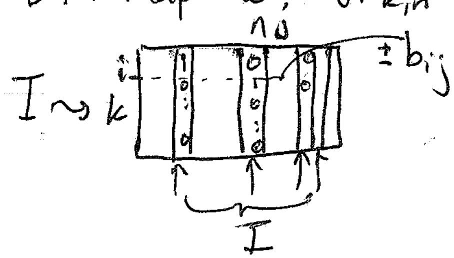
Claim Universal brackets form a 6-parameter family of diagonal quadratic brackets.

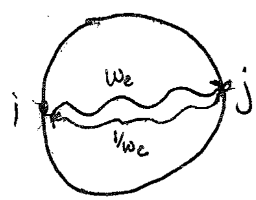


Theorem M.
For any $N \in \mathcal{N}_{I,J}$, β_N induces a 2-parameter family of Poisson brackets on $Mat_{k,m}$ (one can give those explicitly).

Corollary
For $n = 2k$, $I = \cancel{[1, k]}$ $[1,k]=[k]$? one recovers the standard Sklyannin brackets on Mat_{km} among those

BM map to $Gr_{k,n}$





$\gamma_N : \mathcal{E}_N \to Gr_{km}$ path reversal (Postnikov)

Theorem G
1) For any $N \in \mathcal{N}_{I,J}$, γ_N induces a 2-parametric family of Poisson brackets on Gr_{kn}

2) The restriction of this family to the cell $Gr^{I}_{k,n}$ coincides with the family ~~of~~ given by Theorem M.