MSRI
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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean **Email/Phone:** asecelean2@math.unl.edu

Speaker's Name: Dylan Thurston

Talk Title: Cluster Algebra and Triangulated Surfaces

Date: 09 / 07 / 12 **Time:** 10 : 30 am / pm (circle one)

List 6-12 key words for the talk: cluster algebra, triangulation, surface

Please summarize the lecture in 5 or fewer sentances: This lecture discusses connections between cluster algebras and triangulations of surfaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

#3

Cluster Algebras and Triangulated Surfaces

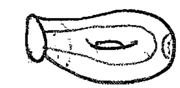
9/7/12

Dylan Thurston

Positivity + Canonical Bases (conjectural)

marked surface (S, M)

Definition A multi-curve C on (S, M) is a ~~†~~ 1-manifold mapped into S, relative to M.



C is simple if it has no self-intersections

Definition Fix a decorated hyperbolic structure on (S,M) with C a multicurve.

$$\lambda(C) = \begin{cases} e^{\ell(c)/2} & C \text{ is an arc} \\ e^{\ell(c)/2} + e^{-\ell(c)/2} & C \text{ is a loop} \\ \prod_i \lambda(C_i) & C \text{ is disconnected } = \underset{\text{connected component}}{\bigcup C_i} \end{cases}$$

$\hookrightarrow$ a function on a decorated Teichmüller space

[why $e^{\ell(c)/2} + e^{-\ell(c)/2}$?

Consider the holonomy of C

$$|tr(A)| = |x + x^{-1}| = \lambda(C).$$

$A \in (P)SL_2(\mathbb{R}) \sim \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, $x = \pm e^{\ell/2}$]

$\ell(C)$ = length of geodesic representative

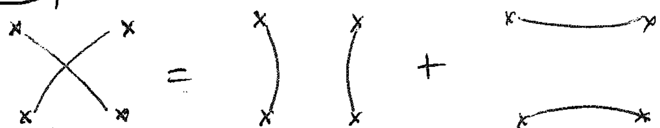
Skein Relation

C multicurve w/crossing

C_+^* C_-^* smoothings

$$\lambda(C) = \pm\lambda(C_+) \pm \lambda(C_-) \quad (\text{us. } +)$$

Example



$$\lambda(C) = \lambda(C_+) + \lambda(C_-)$$

exchange relation

Pf one case:

hol AB hol AB^{-1}

C_1 C_2

C = two loops crossing at chosen crossing

C_+ connected

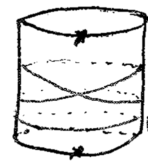
C_-

$\boxed{\text{Lemma}}$
Given $A, B \in SL_2(\mathbb{R})$,
$$tr(A)\,tr(B) = tr(AB) + tr(AB^{-1})$$

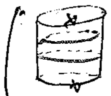
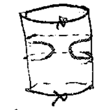
A = holonomy around C_1
B " " C_2

Pf B satisfies it's characteristic equation
$$B^2 - tr(B)\,B + I = 0$$
$$B - tr(B) + B^{-1} = 0$$
$$AB - A\,tr\,B + AB^{-1} = 0 \quad \text{take trace.}$$
□

Example 3



$L^{(2)}$ L

$\lambda(L^{(2)}) = \pm\lambda\Big($  $\Big) \pm \lambda\Big($  $\Big)$

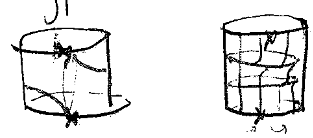
$= \lambda(L)^2 \pm \lambda\Big($  $\Big)$
$$= \lambda(L)^2 - 2$$

In general, $\lambda(L^{(k)}) = T_k(\lambda(L))$
the Chebychev polynomial of first kind.

$\boxed{\text{Lemma}}$
For any multicurve C, $\lambda(C) = \sum \pm \lambda(C_i)$
for some simple multicurves C_i.

$\boxed{\text{Theorem}}$ (Musiker - Schiffler - Williams)
$\{\lambda(C_i) : C_i \text{ a simple multicurve}\}$
form a linear basis for an algebra containing the cluster algebra $A(S,M)$.

For simplicity, we assume <u>no interior points.</u>
Typical elements



$\lambda(C)$ are Laurent polynomials in the cluster variables for any triangulations

(denominators are bounded by intersection numbers

Notions of Positivity

(1) <u>Positive Laurent phenomenon</u>:
If C a simple multicurve, $\lambda(C)$ is a positive Laurent polynomial in a fixed cluster

<u>Theorem</u> (Musiker - Schiffler)
True if no interior marked points.

(2) <u>Positive basis</u> for R
$\exists$ a basis $\{x_i\}$ so that $\forall\, i,j$
$x_i x_j = \sum_k a_{ij}^k x_k$ has ~~positive~~ $a_{ij}^k \geq 0$.

$(2) \Rightarrow (1)$.

$\{\lambda(C) \mid C$ a simple multicurve$\}$
are NOT a positive basis unfortunately.
↳ Recall $\lambda(L^{(2)}) = \lambda(L)^2 - 2$ oh no!

Perhaps this can be fixed!

<u>Definition</u> Let C be a multicurve. C is <u>compatible (with itself)</u> if
(a) no two components of C intersect each other.
(b) Each component has no self intersections unless it is a power of a simple curve ($L^{(k)}$)
(c) No two components are multiple covers of the same curve.

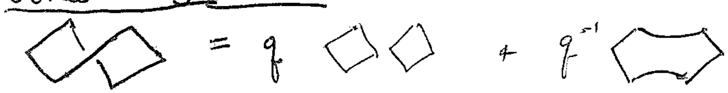
i.e. $C \neq \ \cdots \cup L^{(k)} \cup L^{(\ell)} \uplus \cdots)$

<u>Conjecture</u>
$\{\lambda(C) \mid C$ compatible multicurve$\}$ form a positive basis.

Levels of Cluster algebras

(1) Tropical	surface intersection numbers
(2) Algebraic	$\lambda(E)\lambda(F) = \lambda(A)\lambda(C) + \lambda(B)\lambda(D)$ hyperbolic geometry
(3) Quantum	skein theory of surfaces (Mullen no internal punctures)
(4) Categorified (monoidal)	It would be nice if we could put something here!

<u>Jones Polynomial</u>



expect: get invariants of 4-manifolds