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NOTETAKER CHECKLIST FORM

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Name: Stephen Hermes **Email/Phone:** SRHERMES@BRANDEIS.EDU

Speaker's Name: D. Hernandez

Talk Title: Non-Simply Laced Quantum Affine Algebras : Cluster Algebras.

Date: 11 / 1 / 12 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Quantum Algebras, Monoidal Categorification, Cluster Algebras, Dynkin Diagrams, T-Systems.

Please summarize the lecture in 5 or fewer sentances: The speaker defined monoidal categorifications of cluster algebras and introduced a family of such categorifications defined in terms of quantized enveloping algebras. He showed that certain T-systems arise from this theory.

CHECK LIST

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Non Simply-Laced Quantum Affine Algebras and Cluster Algebras

D. Hernandez

November 1, 2012

Joint with B. Leclerc.

1 Monoidal Categorifications

Consider a cluster algebra $A(\widetilde{B})$ with $\widetilde{B}$ skew-symmetric. Let Q be the associated quiver and $\mathcal{M}$ a monoidal category.

Definition 1.1. $\mathcal{M}$ is a *monoidal categorification* of $A(\widetilde{B})$ is there is a ring isomorphism between $A(\widetilde{B})$ and $K_0(\mathcal{M})$ (the Grothendeick ring) such that through this isomorphism, cluster monomials are the classes of real simple objects of $\mathcal{M}$ (S in $\mathcal{M}$ is *real* if $S \otimes S$ is simple).

Remark 1.2. There is a bijection between cluster variables and the classes of real prime objects (S is *prime* if there is no isomorphism $S \cong S_1 \otimes S_2$ with S_i non-trivial).

Conseqences:

1. For the cluster algebra $A(\widetilde{B})$:
   (a) Every cluster variable of $A(\widetilde{B})$ has a Laurent expansion with positive coefficients with respect to any seed.
   (b) Cluster monomials of $A(\widetilde{B})$ are linearly independent (though there is a general proof of this through additive categorification).
2. For the category $\mathcal{M}$:
   (a) Can compute factorization of simple (real) objects in terms of prime objects.
   (b) Can compute Clebsch-Gordon coefficients of $\mathcal{M}$.

Known Examples:

1. (H., Leclerc '09) Constructed monoidal categorifications by using categories of representations of quantum affine algebras. Got monoidal categorifications of type A, D_4 (bipartite).
2. (Nakajima '09): For all A, D, E types (bipartite).
3. General acyclic (bipartite) case: (almost) obtained monoidal categorification by using perverse sheaves on quiver varieties.
4. (HL '12) By using $U_q(L\mathfrak{g})$ for types A, D (with linear orientation)
5. (Kimura-Qin '12) Generalization of Nakajima's approach to acyclic with general orientation.

Conjecture 1.3 (Still open HL '09)**.** The categories $\mathcal{C}_\ell$ for $\ell \geq 2$ are monoidal categorifications (the above constructions are $\mathcal{C}_1$).

What about non-simply laced quantum affine algebras? At this point, there is no quiver variety theory for these cases.

2 Quantum Affine Algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Let $I = \{1, \ldots, n\}$ be the vertices of the corresponding Dynkin diagram and $r_1, \ldots, r_n$ the corresponding root lengths. Define a Lie algebra $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[\epsilon^{\pm}]$ (Laurent polynomials in ϵ) the *loop algebra* of $\mathfrak{g}$. The quantized enveloping algebra $U_q(L\mathfrak{g})$ is the *quantum loop algebra* (a quotient of a quantum affine algebra: a Drinfel'd-Jimbo quantum group). It known to be a Hopf algebra.

Let $\mathcal{C}$ be the category of finite dimensional representations of $U_q(L\mathfrak{g})$. Simple objects in $\mathcal{C}$ are given by the objects $L(m)$ where

$$m = \prod_{\substack{1 \le i \le n \\ a \in \mathbb{C}^*}} Y_{i,a}$$

(Drinfel'd polynomials). The representation ring is

$$\mathrm{Rep}(U_q(L\mathfrak{g})) = \bigoplus_m \mathbb{Z}[L(m)].$$

Theorem 2.1 (Frenkel-Reshetikhin '98)**.** $\mathrm{Rep}(U_q(L\mathfrak{g}))$ is a commutative polynomial ring generated by the representation $[L(Y_{i,a})]$.

T-systems can be realized in this ring. This suggests it should have something to do with monoidal categorification. To do this we need Kirillov-Reshetikhin modules. For $k \ge 0, a \in \mathbb{C}^*, 1 \le i \le n$ define

$$W_{k,a}^{(i)} = L(Y_{i,a}, Y_{i,aq^{2r_i}}, \ldots, Y_{i,aq^{2(k-1)r_i}})$$

Theorem 2.2 (N04, H06)**.** The $[W_{k,a}^{(i)}]$ satisfy T-systems

$$[W_{k,a}^{(i)}][W_{k,aq^{2r_i}}^{(i)}] = [W_{k+1,a}^{(i)}][W_{k-1,aq^{2r_i}}^{(i)}] + \prod_{i \ne j} [W_{k,aq}^{(j)}]$$

Example 2.3. For $\mathfrak{g} = B_2$, $r_1 = 2, r_2 = 1$.

$$[W_{k,a}^{(1)}][W_{k,aq^4}^{(1)}] = [W_{k+1,a}^{(1)}][W_{k-1,aq^4}^{(1)}] + [W_{2k,aq}^{(2)}]$$
$$[W_{k,a}^{(2)}][W_{k,aq^2}^{(2)}] = [W_{k+1,a}^{(2)}][W_{k-1,aq^2}^{(2)}] + [W_{[(k+1)/2],aq}^{(1)}][W_{[k/2],aq^3}^{(1)}]$$

For the non simply-laced case T-systems were studied in relation to cluster algebras in [Inoue-Iyama-Keller-Kuniba-Nakanishi].

3 Monoidal Subcategories of $\mathcal{C}$

Let $r = \max\{r_i : i \in I\}$ be the *lacing number* of $\mathfrak{g}$.

Definition 3.1. An *upper height* ϕ is a collection $\phi_1, \ldots, \phi_r : \{1, \ldots, n\} \to \mathbb{Z}$ such that:

1. $r_i = 1$ implies $\phi_1(i) = \cdots = \phi_r(i) =: \phi(i)$
2. $c_{ij} c_{ji} = 1$ implies $|\phi_k(i) - \phi_k(j)| = r_i$ for each k
3. $\{\phi_1(j), \ldots, \phi_r(j)\} = \{\phi_1(i) + \epsilon, \ldots, \phi_r(i) + \epsilon + 2 - 2r\}$

Similarly, we define ψ to be a *lower height* if $-\psi$ is upper.

Example 3.2. For B_2, $\phi_1(1) = 2, \phi_2(1) = 4, \phi(2) = 3$ is lower and $\psi_1(1) = 6, \psi_2(1) = 8, \psi(2) = 7$ is upper.

Definition 3.3. Let ϕ be lower and ψ be upper. Define $\mathcal{C}_\phi^\psi$ to be the subcategory of $\mathcal{C}$ of objects whose Jordan-Hölder series involves simple objects of the form $L(m)$ where $m \in \mathbb{Z}[Y_{i,q^\ell}]$ and

$$\ell \in \bigcup_{1 \le k \le \ell} \{\phi_k(i), \phi_k(i) + 2r_i, \ldots, \psi_k(i)\}$$

(Assume $\psi_k(i) = \phi_k k(i) + 2r_i$).

Proposition 3.4. $\mathcal{C}_\phi^\psi$ is a monoidal category.

Theorem 3.5. For each non-simply laced type, there is a non-trivial category $\mathcal{C}_\phi^\psi$ which is a monoidal categorification of a finite type cluster algebra. For $\mathfrak{g}$ of type B_n we get a cluster algebra of type A_{2n}, for C_n get A_{n+1}, F_4 get D_6, for G_2 get A_4.

4 The Proof

1. Consider a family of prime objects. Label by cluster variables in an initial seed.
2. F-polynomials are identities in therms of q-characters in $\mathcal{C}_\phi^\psi$.
3. Use [H '10] to get for $S_1, \ldots, S_n$ in $\mathcal{C}$, $S_1 \otimes \cdots \otimes S_n$ is simple iff $S_i \otimes S_j$ is simple for all $i \neq j$.