Perla Wiener sausages

Sausages w/ circles or squares

classic

Q| Which has bigger expected volume?

→ This talk answers it.

Open Q: stochastic domination?

Thm (Peres–S.)

Let (A_s) open sets in $\mathbb{R}^d$.

Let $(\xi(s))$ std. BM in $\mathbb{R}^d$.

Then $\forall t$: $\mathbb{E} \operatorname{vol} \bigcup_{s \le t} (\xi(s) + A_s) \ge \mathbb{E} \operatorname{vol} \bigcup_{s \le t} (\xi(s) + B(0,r))$

$\forall s$: $\operatorname{vol}(A_s) = \operatorname{vol} B(0,r)$

i.e. smallest for balls.

Motivation

$\Pi = \{X_i\}$ PP in $\mathbb{R}^d$, w/ intensity λ.

Now $(\xi_i(t))$ ind. BM's, at time t the position of i^{th} node $X_i + \xi_i(t)$

think of particles as nodes of a mobile detection network.

$$T_{det} = \inf\{ t \geq 0 : \exists i \ X_i + \xi_i(t) \in B(0,r) \}$$

$$T^f_{det} = \inf\{ t \geq 0 : \exists i \ X_i + \xi_i(t) \in B(f(t),r) \}.$$

How is this related to volume of Wiener sausage?

Classical result

$$\mathbb{P}(T^f_{det} > t) = e^{-\lambda \mathbb{E}[\operatorname{vol} \bigcup_{s\le t} (\xi(s) + B(f(s),r))]}$$

an ineq. for vol.s is same as ineq. for prob.s of going undetected.

Proof

$$\Phi = \{ X_i \in \Pi : \exists s \le t \ X_i + \xi_i(s) \in B(f(s),r) \}$$

Φ : thinned PP.

$$\Lambda(x) = \lambda \mathbb{P}(x \in \bigcup_{s \le t} B(\xi(s) + f(s), r))$$

$$\mathbb{P}(T^f_{det} > t) = \mathbb{P}(\Phi(\mathbb{R}^d) = 0) = e^{-\int_{\mathbb{R}^d} \Lambda(x)dx} \qquad \square$$

Thm (Peres - Sinclair - S - Stauffer) 3

f cts.

$\underline{d=1}$: $\forall t$ $\mathbb{E}\,\mathrm{vol}\bigcup_{s\le t} B(\xi(s)+f(s),r) \geq \mathbb{E}\,\mathrm{vol}\bigcup_{s\le t} B(\xi(s),r)$

Best strategy for non-detection : stay in place.

$d=2$ — — — $\geq (1-o(1))\cdot$ — — — as $t\to\infty$

$d\geq 3$ — — — $\geq c(d)\cdot$ — — —

$\mathbb{P}(\tau_D > t) =$

using rearrangement ineq.s.

$\mathbb{P}(T_{det}^f > t) = \mathbb{P}(\forall s=0,\dots,t \ \nexists i \ \ X_i + \zeta_i(s) \in B(f(s),r))$

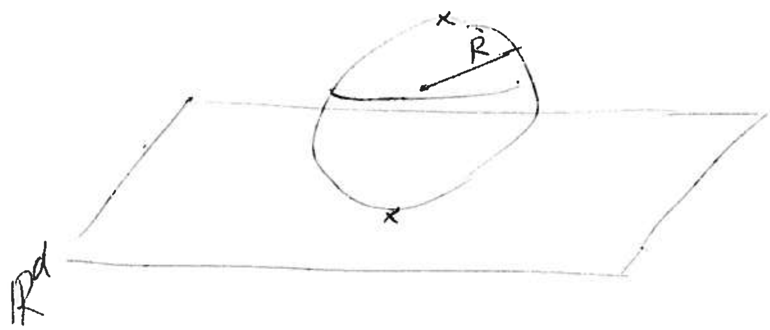
$A_s = B(f(s),r)$

$= \mathbb{E}[\mathbb{P}(\forall s=0,\dots,t \ \zeta(s)\notin A_s)^N)$

$= \int\cdots\int \prod p(x_1,\dots,x)\ \mathbb{1}[x_i\in A_i^c]\,d\underline{x}$

in $\mathbb{R}^d$: complement of a ball is not a ball

in a sphere : complement of a cup is a cup



$A_1, \ldots, A_t$

$A_1^*, \ldots, A_t^*$

then apply Almut + Schmuckelschlager's result
on rearrangements.

Pure motivation for studying BM w/drift.

E.g. H dim. of BM w/ drift is
$\geq$ H dim of BM.

<u>Pascal's principle</u> : staying put minimizes prob. of detection