Calculus for Nonlinear Spectral Gaps (Manor Mendel)

Poincare ineq for graphs

Let A be symm. stochastic $n \times n$ matrix

Let $\gamma = \inf \gamma > 0$ st

$$\forall\ f : \{1, \dots, n\} \to \mathbb{R} \ (\text{or } \ell_2)$$

$$\frac{1}{n^2} \sum_{i,j} \| f(i) - f(j) \|^2 \leq \gamma \frac{1}{n} \sum_{i,j} A_{ij} \| f(i) - f(j) \|^2$$

Claim: $\gamma(A) = \frac{1}{1 - \lambda_2(A)}$.

Thm (Linial, London, Rabinovich; Matousek; Gromov):

Expanders do not embed well in ℓ_2.

Def: Define $\gamma(A, X)$ by replacing ℓ_2 with X in the def^n of γ.

The idea is to show that expanders do not embed well in ℓ_2.

Def: A Banach space is uniformly convex if

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \frac{\|x+y\|}{2} ;\ \|x\| = \|y\| = 1,\ \|x - y\| \geq \varepsilon \right\}.$$

is positive $\forall\ \varepsilon > 0$.

Q: Kasparov-Yu: are there expanders wrt uniformly convex Banach spaces?

Thm (Lafforgue): Yes, but the proof was complicated.

Reingold-Vadhan-Wigderson gave a combinatorial construction of expanders using a "zig-zag" product:

Start with a nice enough H d-reg on d^{2t} verts.

$G_0 = H^2$

$\vdots$

$G_{i+1} = G_i^t \text{ ⓩ } H.$

$$\gamma(G \text{ ⓩ } H) \leq \gamma(H) \cdot \gamma(H^2)$$

$$\gamma(G^t) = \frac{1}{1 - \lambda_2(G)^t}$$

$$\lesssim \max\{1, \gamma(G)/t\}$$

$$\Rightarrow \gamma(G_{i+1}) \lesssim C \max\left\{1, \frac{\gamma(G_i)}{t}\right\} \cdot \gamma(H)^2$$

$\leq$ uniform constant if $\gamma(H) \leq \sqrt{t}$.

On a general metric space,

$$\gamma(G \textcircled{z} H, X) \leq \gamma(G, X) \cdot \gamma(H, X)^2.$$

More problematic is $\gamma(G^t)$: there are some metric spaces on which this doesn't decay well enough.

Main results:

- a new proof of Lafforgue's thm
- examples of $\mathcal{E}, G, X$: $G \hookrightarrow X$, $\gamma(\mathcal{E}, X) < c$

Lipschitz extensions

$$\begin{tikzcd} F: Y \arrow[rr, dashed] & & X \\ & Z \arrow[ul, "id"] \arrow[ur] & \\ f: & & \end{tikzcd}$$

want F s.t. $\|F\|_{Lip} \leq C_{X,Y} \|f\|_{Lip}$

K. Ball: if X is ℓ_p, $1 < p \leq 2$ (more generally, Markov co-type 2

Y has Markov type 2

then you can extend

Thm: Hadamard spaces have Markov co-type 2.

Examples of Hadamard spaces: Euclidean, hyperbolic, large girth graphs

Def: X is P-barycentric if $\forall$ bounded prob meas μ on X, $\exists$ $b(\mu)$ s.t.

1) $\forall \mu, \sigma$ $\quad d_X(b(\mu), b(\sigma)) \leq c W_1(\mu, \sigma)$ ↙ wasserstein 1-distance

2). $\forall \mu$, $a \in X$,

$$d_X(a, b(\mu))^p + c \int d_X(b(\mu), y)^p \, d\mu(y) \leq \int d_X(a, y)^p \, d\mu(y)$$

Turns out Hadamard spaces are barycentric w/ $p=2$. (this can be taken as the definition).

For a random variable Z in a p-barycentric Banach space, $\|\mathbb{E} Z\|^p + k^{-p}\, \mathbb{E}\|Z - \mathbb{E} Z\|^p \le \mathbb{E}\|Z\|^p$.

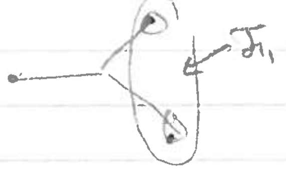
From this, we derive a martingale inequality:

take a filtration $\mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_m$.

M_n a martingale if

$M_{n-1} = b(M_n \mid \mathcal{F}_{n-1})$.

Example:



$\mathcal{F}_0$ trivial, $\mathcal{F}_2$ is everything

$$\boxed{M_0 \neq b(M_2 \mid \mathcal{F}_0)} \quad !$$

<u>Martingale cotype</u>: if $\mathbb{E}\, d_X(M_m, M_0)^p \ge c_0 \sum_{i=0}^{m-1} \mathbb{E}\, d_X(M_i, M_{i+1})$ then we say X has martingale cotype p.

Recall <u>Markov type</u>: X has Markov type 2 if for all stochastic matrix A, $x_1, \ldots, x_n$,

$$\sum (A^m)_{ij}\, d_X(x_i, x_j)^2 \le M \sum A_{ij}\, d_X(x_i, x_j)^2$$

<u>Markov cotype</u>: $\forall\ x_1, \ldots, x_n\ \exists\ y_1, \ldots, y_n$ s.t.

$$\sum (A^m)_{ij}\, d(x_i, x_j)^2$$

$$\ge \max\Big\{ m \sum_{i,j} A_{ij}\, d_X(y_i, y_j)^2,\ \sum d(x_i, y_i)^2 \Big\}.$$