The Number of Entangled Clusters

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Outline

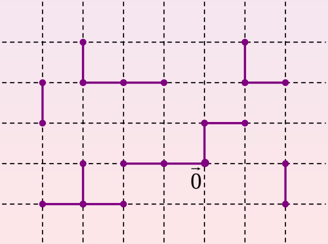
- Percolation and Entanglement Percolation
- The Conjecture of Grimmett & Holroyd
- Related Problems
- Idea of the Proof

Bond Percolation in $\mathbb{Z}^d$

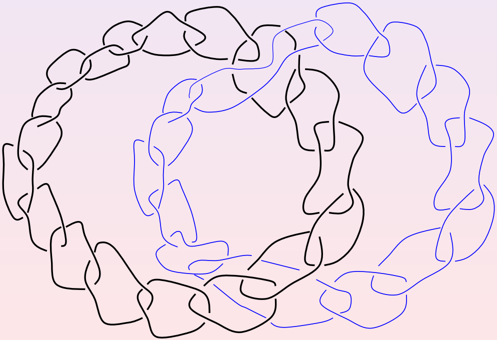
- Each edge is "open" with prob. p and "closed" with prob. $1-p$
- Let $C(conn)$ = connected component of open edges containing $\vec{0}$
- Hammersley + Broadbent (1957-59) proved

$$\exists\, p_c \in (0,1) \text{ such that } P_p(|C(conn)| = \infty) \begin{cases} = 0 & p < p_c \\ > 0 & p > p_c \end{cases}$$

- For $p > p_c$, there is a (unique) infinite connected open component a.s.



- A model for (random) polymer networks
- $p > p_c$: gelation (infinite network)
- $p < p_c$: may get large (possibly infinite) network of small polymers, topologically linked (entangled)
- entangled: can't separate by deformed sphere



- $C(\mathcal{E})$ = (maximal) entangled component of open edges containing $\vec{0}$
- $P_p(|C(\mathcal{E})| = \infty) \begin{cases} = 0 & p < p_E \\ > 0 & p > p_E \end{cases}$ $\qquad p_E \leq p_c$
- Monte Carlo simulations of Kantor & Hassold (1988) suggest $p_c - p_e \approx 10^{-7}$.

Theorem (Grimmett & Holroyd 2000-2002, Aizenmann & Grimmett 1991)

$$\frac{1}{15616} \leq p_E < p_c$$

Theorem (Grimmett & Holroyd 2010)

$$p_E \geq \mu_3^{-2} > \frac{1}{23}$$

where μ_3 is the connective constant for self-avoiding walks in $\mathbb{Z}^3$.

- Let a_N = number of connected graphs with N edges in $\mathbb{Z}^3$(modulo translation)
- Let e_N = number of entangled graphs with N edges in $\mathbb{Z}^3$(modulo translation)

Classical Theorem (e.g. Klarner 1967)

$$\lambda := \lim_{N\to\infty} a_N^{1/N} < \infty$$

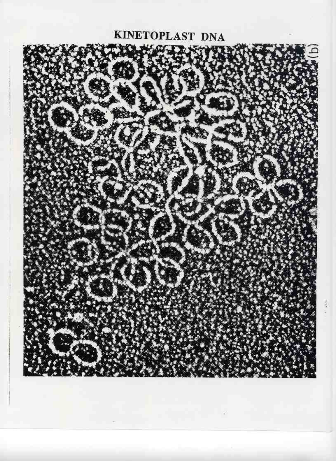
Theorem (Grimmett and Holroyd 2000)

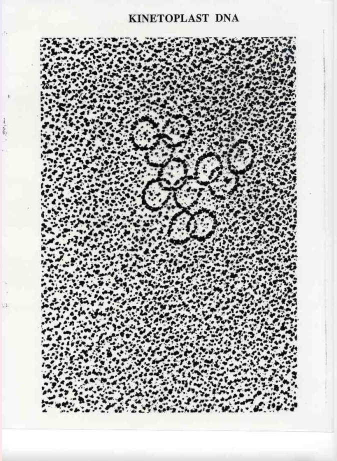
$$e_N \leq e^{o(N \log N)}$$

Conjecture (Grimmett and Holroyd 2000)

$$\lambda_e := \lim_{N\to\infty} e_N^{1/N} < \infty?$$

As the self-avoiding walk models linear polymers, the self-avoiding polygon models ring polymers, and a_N corresponds to branched polymers, so e_N corresponds to networks of entangled polymers.





Theorem (Atapour + Madras)

$$\lambda_e \leq 4\lambda^2$$

Corollary 1

$$p_E \geq \frac{1}{\lambda_e} > \frac{1}{597}$$

Corollary 2

$$P_p(|C(\mathcal{E})| \geq n) \leq e^{-cN} \text{ for } p < \frac{1}{\lambda_e}$$

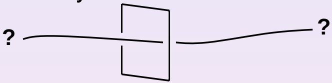
Proof of Corollaries.

$$\begin{aligned} P_p(|C(\mathcal{E})| \geq n) &\leq \sum_{\text{finite entangled } G:\ \vec{0}\in G,\ |G|\geq n} P_p(G \subset C(\mathcal{E})) \\ &\leq \sum_{N\geq n} e_N p^N \approx \frac{(\lambda_e p)^n}{1-\lambda_e p} \end{aligned}$$

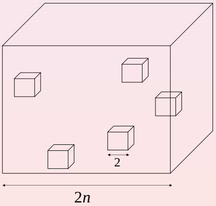
which decays exponentially if $p < \frac{1}{\lambda_e}$ □

Holroyd proved $p_E > 0$ via dual percolation of a surface around 0.
Why is this problem harder than usual percolation?

- $\{|C(\mathcal{E})| \geq n\}$ is not a cylinder event



- ("similar" problem): Let $CC_N =$ number of "caged clusters" with N edges (geometric trap)
- $N/2$ edges in surface of the cube, diam $\approx \sqrt{N}$, vol $\approx N^{3/2}$
- Scatter $(N/2)/12$ unit cubes
- $CC_N \geq \binom{o(N^{3/2})}{o(N)} \approx (N^{3/2})^N$



Recall $a_N =$ number of N-edge connected graphs $\approx \lambda^N$.

Proposition (Kesten)

$$\lambda \leq \frac{5^5}{4^4} \approx 12.2$$

Proof.

$$\begin{aligned} 1 \geq P_p(|C(conn)| = n) &= \sum_{A \ni 0, |A| = n} p^n (1-p)^{\partial A} \\ &\geq a_n p^n (1-p)^{4n+6} \end{aligned}$$

Note that $\partial A \leq 4n + 6$ (worst case: line).

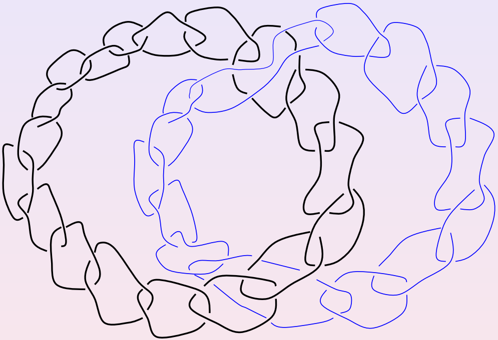
$$\therefore 1 \geq \lambda p (1-p)^4, \qquad \text{i.e.} \quad \frac{1}{p(1-p)^4} \geq \lambda \, .$$

Optimize: $p = \frac{1}{5}$. □

Intuition for proof that $\lambda_e < \infty$

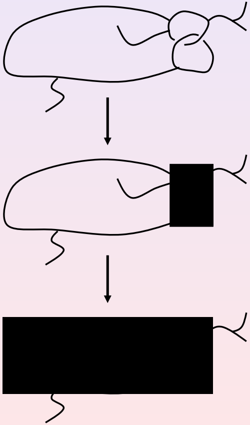
- Let G be a finite entangled graph with $|G| = N$.
- Assume $\exists\, A \subset \mathbb{Z}^3$ such that $G \cup A$ is connected and $|A| \leq t(N)$ (some function of N only). Then $e_N \leq 2^{N+t(N)} a_{N+t(N)} \approx (2\lambda)^{N+t(N)}$. Goal: Show $t = O(N)$
- $g_1, \cdots g_k$: connected components of G
- $\forall i,\ \exists j$ such that $conv(g_i) \cap g_j \neq \emptyset$
- Can connect g_i to g_j with $\leq |g_i|$ (diam of $conv(g_i)$) edges
- Note that $\exists$ cycle in g_i: hence can do it with $\leq \frac{|g_i|}{2}$ edges
- Not finished!
- Next level: diam $\leq \frac{|braceled|}{4}$
- $t(N) \leq \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \cdots \leq N$ (weak on details; right answer.)
- $\therefore e_N \leq (2\lambda)^{2N}$

The Number of Entangled Clusters



The Number of Entangled Clusters

- Convex hull too crude
- Block Cluster: Boxes + Edges + Vertices



Lemma

g connected $\Rightarrow$ any coordinate plane cutting a box of $BC(g)$ cuts ≥ 2 edges of g

Definition

$$g_i \searrow g_j \text{ if } g_j \cap BC(g_i) \neq \emptyset$$

$$\Downarrow$$

- Can connect g_i to g_j by at most diam(box) edges
- Each unit of diameter $\leftrightarrow$ 2 edges of g_i in box

Algorithm: Let G be a finite entangled graph. To connect G,

- Colour all edges of G green
- If G is not connected, $\exists\ g_1 \searrow g_2 \searrow \cdots \searrow g_q \searrow g_1$
- Connect g_i to g_{i+1} inside a box
- Each new edge is Green
- For each new edge, recolour two old Green edges to Red

Theorem

This can be done.

$$\Downarrow$$

at most N edges need to be added.