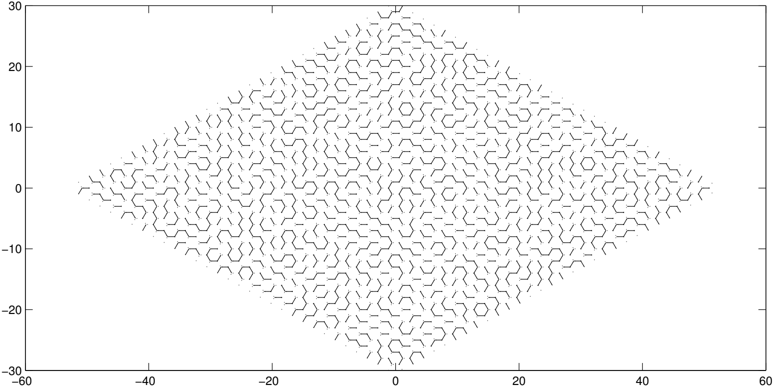
1-2 Model, Dimers and Clusters

Zhongyang Li
University of Cambridge

January 12, 2012

1-2 Model

- Probability measure on subgraphs $\omega = (V, E_\omega)$ of hexagonal lattice $\mathbb{H} = (V, E)$, such that each vertex has 1 or 2 incident edges in ω



Probability measure

- A local configuration at vertex v is a subset of incident edges of v

Probability measure

- A local configuration at vertex v is a subset of incident edges of v
- If a vertex v has degree k (k incident edges), the total number of local configurations at v is 2^k.

Probability measure

- A local configuration at vertex v is a subset of incident edges of v
- If a vertex v has degree k (k incident edges), the total number of local configurations at v is 2^k.
- A weight (positive number) is assigned to each local configuration

Probability measure

- A local configuration at vertex v is a subset of incident edges of v
- If a vertex v has degree k (k incident edges), the total number of local configurations at v is 2^k.
- A weight (positive number) is assigned to each local configuration
- $\omega|_v$: the local configuration obtained when looking at a configuration ω at a vertex v, or, $\omega|_v = \Gamma(v) \cap E_\omega$

Probability measure

- A local configuration at vertex v is a subset of incident edges of v
- If a vertex v has degree k (k incident edges), the total number of local configurations at v is 2^k.
- A weight (positive number) is assigned to each local configuration
- $\omega|_v$: the local configuration obtained when looking at a configuration ω at a vertex v, or, $\omega|_v = \Gamma(v) \cap E_\omega$
- $\Gamma(v)$: the set of incident edges of v in $\mathbb{H}$

Probability measure

- A local configuration at vertex v is a subset of incident edges of v
- If a vertex v has degree k (k incident edges), the total number of local configurations at v is 2^k.
- A weight (positive number) is assigned to each local configuration
- $\omega|_v$: the local configuration obtained when looking at a configuration ω at a vertex v, or, $\omega|_v = \Gamma(v) \cap E_\omega$
- $\Gamma(v)$: the set of incident edges of v in $\mathbb{H}$
- $P(\omega)$ is defined to be proportional to the product of weights of its local configurations at all vertices.

$$P(\omega) = \frac{\prod_{v \in V} w(\omega|_v)}{Z}$$

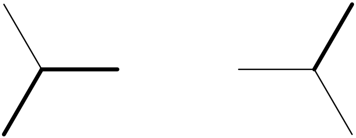
Probability measure

- A local configuration at vertex v is a subset of incident edges of v
- If a vertex v has degree k (k incident edges), the total number of local configurations at v is 2^k.
- A weight (positive number) is assigned to each local configuration
- $\omega|_v$: the local configuration obtained when looking at a configuration ω at a vertex v, or, $\omega|_v = \Gamma(v) \cap E_\omega$
- $\Gamma(v)$: the set of incident edges of v in $\mathbb{H}$
- $P(\omega)$ is defined to be proportional to the product of weights of its local configurations at all vertices.

$$P(\omega) = \frac{\prod_{v \in V} w(\omega|_v)}{Z}$$

- Z is a normalizing constant called the partition function.

Examples of local configurations



101 001

Signature

- A signature r_v of a vertex v is a vector labeled by all the possible local configurations at v.

Signature

- A signature r_v of a vertex v is a vector labeled by all the possible local configurations at v.
- r_v is a vector of dimension $2^{deg(v)}$.

Signature

- A signature r_v of a vertex v is a vector labeled by all the possible local configurations at v.
- r_v is a vector of dimension $2^{deg(v)}$.
- The entry of r_v at a local configuration is the weight of the local configuration.

Signature

- A signature r_v of a vertex v is a vector labeled by all the possible local configurations at v.
- r_v is a vector of dimension $2^{deg(v)}$.
- The entry of r_v at a local configuration is the weight of the local configuration.
- In this talk, we consider 1-2 model whose signature has the following form

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 0 & a & b & c & c & b & a & 0 \end{array},$$

Signature

- A signature r_v of a vertex v is a vector labeled by all the possible local configurations at v.
- r_v is a vector of dimension $2^{deg(v)}$.
- The entry of r_v at a local configuration is the weight of the local configuration.
- In this talk, we consider 1-2 model whose signature has the following form

$$\begin{pmatrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 0 & a & b & c & c & b & a & 0 \end{pmatrix},$$

- Each configuration and its complement have the same weight.

Signature

- A signature r_v of a vertex v is a vector labeled by all the possible local configurations at v.
- r_v is a vector of dimension $2^{deg(v)}$.
- The entry of r_v at a local configuration is the weight of the local configuration.
- In this talk, we consider 1-2 model whose signature has the following form

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 0 & a & b & c & c & b & a & 0 \end{array},$$

- Each configuration and its complement have the same weight.
- The model is uniform if $a = b = c = 1$.

History

- Computer scientists Schwartz and Bruck (2008) proposed the uniform 1-2 model (not-all-equal relation), as a graphical model whose partition function (total number of possible configurations) can be computed by computing determinants via holographic algorithm.

History

- Computer scientists Schwartz and Bruck (2008) proposed the uniform 1-2 model (not-all-equal relation), as a graphical model whose partition function (total number of possible configurations) can be computed by computing determinants via holographic algorithm.
- We introduced a generalized algorithm in a previous paper (2011), which could solve more models including the 1-2 model defined above.

History

- Computer scientists Schwartz and Bruck (2008) proposed the uniform 1-2 model (not-all-equal relation), as a graphical model whose partition function (total number of possible configurations) can be computed by computing determinants via holographic algorithm.
- We introduced a generalized algorithm in a previous paper (2011), which could solve more models including the 1-2 model defined above.
- However, the holographic algorithm, although very general and beautiful, is not an efficient way to solve the 1-2 model.

Correspondence: 1-2 model and dimers

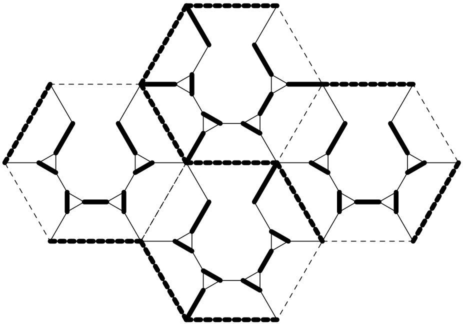


Figure: 1-2 Model and Dimers

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).
- Dimer model is defined on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ we constructed.

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).
- Dimer model is defined on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ we constructed.
- $V \subset V_\Delta$.

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).
- Dimer model is defined on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ we constructed.
- $V \subset V_\Delta$.
- For $v \in V$, incident edges of v in E_Δ are bisectors of the angles of $\mathbb{H}$ at v.

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).
- Dimer model is defined on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ we constructed.
- $V \subset V_\Delta$.
- For $v \in V$, incident edges of v in E_Δ are bisectors of the angles of $\mathbb{H}$ at v.
- On each face of $\mathbb{H}$, draw a small hexagon with vertices incident to the bisector edges.

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).
- Dimer model is defined on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ we constructed.
- $V \subset V_\Delta$.
- For $v \in V$, incident edges of v in E_Δ are bisectors of the angles of $\mathbb{H}$ at v.
- On each face of $\mathbb{H}$, draw a small hexagon with vertices incident to the bisector edges.
- Remove the top most edge of each small hexagon.

Construction of the dimer graph

- 1-2 model is defined on the hexagonal lattice $\mathbb{H} = (V, E)$ (the dashed graph).
- Dimer model is defined on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ we constructed.
- $V \subset V_\Delta$.
- For $v \in V$, incident edges of v in E_Δ are bisectors of the angles of $\mathbb{H}$ at v.
- On each face of $\mathbb{H}$, draw a small hexagon with vertices incident to the bisector edges.
- Remove the top most edge of each small hexagon.
- Change each degree-3 vertex of each small hexagon by a triangle.

Correspondence: 1-2 model and dimers

- A bisector edge is present if and only if the two edges of the angle have the same configuration.

Correspondence: 1-2 model and dimers

- A bisector edge is present if and only if the two edges of the angle have the same configuration.
- Once configurations on all bisector edges are known, there is a unique extension to a perfect matching on $\mathbb{H}_{\Delta}$.

Correspondence: 1-2 model and dimers

- A bisector edge is present if and only if the two edges of the angle have the same configuration.
- Once configurations on all bisector edges are known, there is a unique extension to a perfect matching on $\mathbb{H}_\Delta$.
- A 1-2 model configuration and its complement correspond to the same perfect matching.

Correspondence: 1-2 model and dimers

- A bisector edge is present if and only if the two edges of the angle have the same configuration.
- Once configurations on all bisector edges are known, there is a unique extension to a perfect matching on $\mathbb{H}_{\Delta}$.
- A 1-2 model configuration and its complement correspond to the same perfect matching.
- Given appropriate edge weights to $\mathbb{H}_{\Delta}$, such a correspondence is measure-preserving.

Clusters

- Local configurations 001 and 110 have weight a, we call them a-configurations.

Clusters

- Local configurations 001 and 110 have weight a, we call them a-configurations.
- An a-cluster is a connected set of vertices, all of which have a-configurations.

Clusters

- Local configurations 001 and 110 have weight *a*, we call them *a*-configurations.
- An *a*-cluster is a connected set of vertices, all of which have *a*-configurations.
- In fact, for vertices in a single *a*-cluster, either all of them have the configuration 001, or all of them have the configuration 110, because the configuration 001 and 110 cannot appear on a pair of neighboring vertices.

Existence of Phase Transition

Theorem

Fix $b, c > 0$, and use a large torus to approximate the infinite periodic graph. When a is sufficiently small, almost surely there is no infinite a-clusters; when a is large, the probability of the existence of infinite a-clusters is strictly positive, and the number of infinite a-clusters is at most one almost surely.

Sketch of Proof: 1

- P_a, the probability that a-configuration appear at a vertex, has an exact integral formula.

Sketch of Proof: 1

- P_a, the probability that a-configuration appear at a vertex, has an exact integral formula.
- P_a is continuous at 0 and ∞, moreover,

$$\lim_{a\to 0} P_a = 0$$
$$\lim_{a\to\infty} P_a = 1$$

Sketch of Proof: 1

- P_a, the probability that a-configuration appear at a vertex, has an exact integral formula.
- P_a is continuous at 0 and ∞, moreover,

$$\begin{aligned} \lim_{a\to 0} P_a &= 0 \\ \lim_{a\to\infty} P_a &= 1 \end{aligned}$$

- Using a large deviation argument and results of determinantal processes, we prove that if V_0 is an arbitrary set of vertices,

$$P(\textit{All the vertices in } \mathrm{V}_0 \text{ have } a\text{-configurations}) \leq (P_a)^{|V_0|}$$

Sketch of Proof: 2

- $\mathcal{A}_k$: the number of animals (connected set of vertices) of size k including the origin.

Sketch of Proof: 2

- $\mathcal{A}_k$: the number of animals (connected set of vertices) of size k including the origin.
- $\mathcal{A}_k \leq t^k$, where $t > 0$ is a constant independent of k.

Sketch of Proof: 2

- $\mathcal{A}_k$: the number of animals (connected set of vertices) of size k including the origin.
- $\mathcal{A}_k \leq t^k$, where $t > 0$ is a constant independent of k.
- $\mathcal{U}_k$: the set of animals of size k including the origin

Sketch of Proof: 2

- $\mathcal{A}_k$: the number of animals (connected set of vertices) of size k including the origin.
- $\mathcal{A}_k \leq t^k$, where $t > 0$ is a constant independent of k.
- $\mathcal{U}_k$: the set of animals of size k including the origin
-

$$
\begin{aligned}
& P(an\ infinite\ a-cluster\ appears\ at\ the\ origin) \\
\leq \quad & \lim_{k\to\infty} \sum_{\{s_k : s_k \in \mathcal{U}_k\}} P(all\ vertices\ in\ s_k\ have\ a-configurat \\
\leq \quad & \lim_{k\to\infty} (tP_a)^k
\end{aligned}
$$

$= 0 \text{if } tP_a < 1$

Sketch of Proof: 2

- $\mathcal{A}_k$: the number of animals (connected set of vertices) of size k including the origin.
- $\mathcal{A}_k \leq t^k$, where $t > 0$ is a constant independent of k.
- $\mathcal{U}_k$: the set of animals of size k including the origin
-

$$
\begin{aligned}
& P(an\ infinite\ a-cluster\ appears\ at\ the\ origin) \\
\leq\ & \lim_{k\to\infty} \sum_{\{s_k : s_k \in \mathcal{U}_k\}} P(all\ vertices\ in\ s_k\ have\ a-configurat \\
\leq\ & \lim_{k\to\infty} (tP_a)^k
\end{aligned}
$$

$$= 0 \text{if } tP_a < 1$$

- We deduce that when a is small, almost surely there is no infinite a-clusters.

Sketch of Proof: 3

- The boundary vertices of a connected set of vertices V_0 are those adjacent to vertices in V_0, but not in V_0 themselves.

Sketch of Proof: 3

- The boundary vertices of a connected set of vertices V_0 are those adjacent to vertices in V_0, but not in V_0 themselves.
- $\mathcal{S}_p$: the event that all the vertices in a box T centered at the origin have a-configurations, the number of boundary vertices of the box is no less than p.

Sketch of Proof: 3

- The boundary vertices of a connected set of vertices V_0 are those adjacent to vertices in V_0, but not in V_0 themselves.
- $\mathcal{S}_p$: the event that all the vertices in a box T centered at the origin have a-configurations, the number of boundary vertices of the box is no less than p.
- B_q: the set of boundary vertices of a set $\tilde{T}$, such that $T \subset \tilde{T}$, and the number of boundary vertices of $\tilde{T}$ is q .

Sketch of Proof: 3

- The boundary vertices of a connected set of vertices V_0 are those adjacent to vertices in V_0, but not in V_0 themselves.
- $\mathcal{S}_p$: the event that all the vertices in a box T centered at the origin have a-configurations, the number of boundary vertices of the box is no less than p.
- B_q: the set of boundary vertices of a set $\tilde{T}$, such that $T \subset \tilde{T}$, and the number of boundary vertices of $\tilde{T}$ is q .
-

$$
\begin{aligned}
& P(an\ infinite\ a-cluster\ appears\ at\ theorigin) \\
\geq\ & P(an\ infinite\ a-cluster\ appears\ at\ the\ origin|\mathcal{S}_p)P(\mathcal{S}_p) \\
=\ & [1-P(no\ infinite\ a-clusters\ at\ the\ origin|\mathcal{S}_p)]P(\mathcal{S}_p) \\
\geq\ & [1-\sum_{q\geq p}\sum_{B_q}P(none\ of\ vertices\ in\ B_q\ have\ a-configurations|\mathcal{S}_p) \\
& \cdot P(\mathcal{S}_p)
\end{aligned}
$$

Sketch of Proof: 4

-
$$\begin{aligned} & P(an\ infinite\ a-cluster\ appears\ at\ the\ origin) \\ \geq\ & P(\mathcal{S}_p) - \sum_{q\geq p}\sum_{B_q} P(none\ of\ vertices\ in\ B_q\ have\ a-configurations \\ \geq\ & P(\mathcal{S}_p) - (\beta \max\{P_b, P_c\})^p \end{aligned}$$

Sketch of Proof: 4

- $$\begin{aligned} & P(an\ infinite\ a-cluster\ appears\ at\ the\ origin) \\ \geq\ & P(\mathcal{S}_p) - \sum_{q\geq p}\sum_{B_q} P(none\ of\ vertices\ in\ B_q\ have\ a-configuration \\ \geq\ & P(\mathcal{S}_p) - (\beta \max\{P_b, P_c\})^p \end{aligned}$$
- $\beta > 0$ is a constant independent of p

Sketch of Proof: 4

- $$\begin{aligned} & P(an\ infinite\ a-cluster\ appears\ at\ the\ origin) \\ \geq\ & P(\mathcal{S}_p) - \sum_{q\geq p}\sum_{B_q} P(none\ of\ vertices\ in\ B_q\ have\ a-configurations \\ \geq\ & P(\mathcal{S}_p) - (\beta \max\{P_b, P_c\})^p \end{aligned}$$
- $\beta > 0$ is a constant independent of p
- We deduce that when a is large, the probability that an infinite a-cluster appears at the origin is strictly positive.