Six- and Eight-Vertex models on their combinatorial line

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Spin chains

A standard model for magnetism in one dimension:

$$H = -\frac{1}{2}\sum_{i=1}^{L}\left(J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z\right)$$

where the $\sigma_i^{x,y,z}$ are Pauli matrices acting on the $i^{\rm th}$ factor of $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$. (we assume periodic boundary conditions, i.e., $L+1\equiv 1$)

- the XXX chain: $J_x = J_y = J_z$ [Heisenberg, 1928; Bethe, 1931]
- the XXZ chain: $J_x = J_y \neq J_z$ [Yang and Yang, 1966]
- the XYZ chain: $J_x \neq J_y \neq J_z$ [Baxter, 1973]

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Quantum integrability

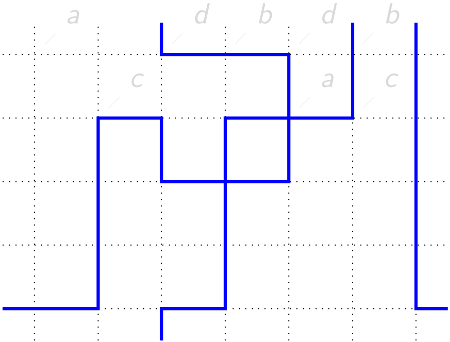
The XXX/XXZ/XYZ Hamiltonian commutes with a one-parameter family of operators on $\mathcal{H}$ called transfer matrices:

$$[T(u), T(v)] = 0 \qquad [H, T(u)] = 0$$

For the XXX/XXZ case, $u \in \mathbb{C}$, whereas in the XYZ case, u lives on an elliptic curve.

Six- and Eight-vertex models

A configuration of the eight-vertex model:



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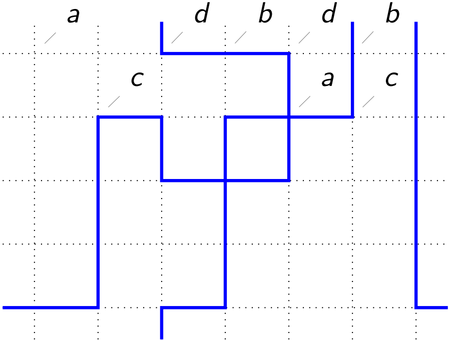

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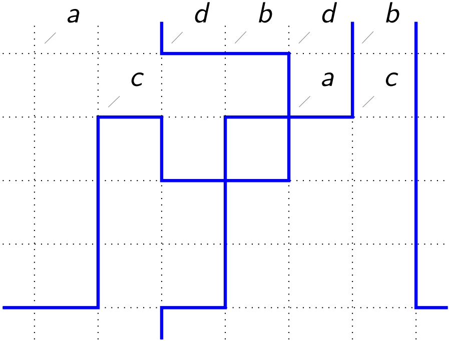

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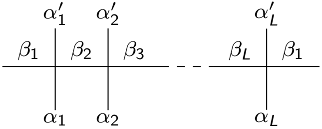
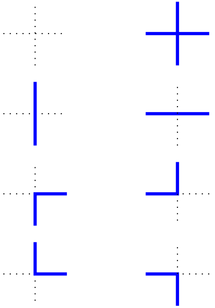
Transfer matrix

The operator $T(u)$ has the meaning of row-to-row transfer matrix on the two-dimensional square lattice: $\mathbb{C}^2 = \langle | = +, \vdots = - \rangle$

$$\langle \alpha' | \, T(u) \, | \alpha \rangle = \sum_{\beta_1,\ldots,\beta_L = -, \cdots}$$





$$a(u) = \vartheta_4(2\eta)\vartheta_4(u)\vartheta_1(u+2\eta)$$

$$b(u) = \vartheta_4(2\eta)\vartheta_1(u)\vartheta_4(u+2\eta)$$

$$c(u) = \vartheta_1(2\eta)\vartheta_4(u)\vartheta_4(u+2\eta)$$

$$d(u) = \vartheta_1(2\eta)\vartheta_1(u)\vartheta_1(u+2\eta)$$

Ground state

We are particularly interested in the ground state of the spin chain, that is the eigenvector of H corresponding to its lowest eigenvalue. In a certain range of u it coincides with the largest eigenvalue of $T(u)$.

If L is odd the ground state is two-fold degenerate, but we make it unique by imposing $\prod_{i=1}^{L} \sigma_i^x = 1$.

In general, using the Bethe Ansatz (or other related methods), one can obtain explicit formulae for the ground state only in the thermodynamic limit where $L \to \infty$.

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Conjecture (Baxter, '72,'73,'89; Stroganov, '99)

Suppose L is odd, and

$$J_x J_y + J_x J_z + J_y J_z = 0$$

Then the ground state eigenvalue is

$$E_0 = -\frac{L}{2}(J_x + J_y + J_z)$$

Combinatorial line

Parameterize (up to normalization)

$$J_x = 1+\zeta \qquad J_y = 1-\zeta \qquad J_z = \Delta$$

then $J_x J_y + J_x J_z + J_y J_z = 0$ equivalent to $\Delta = \frac{\zeta^2-1}{2}$.

Further,

$$\zeta = \left(\frac{\vartheta_1(2\eta,p)}{\vartheta_4(2\eta,p)}\right)^2 \qquad \Delta = \frac{\vartheta_4^2(0)}{\vartheta_2(0,p)\vartheta_3(0,p)} \frac{\vartheta_2(2\eta,p)\vartheta_3(2\eta,p)}{\vartheta_4^2(2\eta,p)}$$

then $J_x J_y + J_x J_z + J_y J_z = 0$ equivalent to $\eta = \pi/3$.

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XXZ/6-vertex case

The XXZ/6-vertex case corresponds to $\zeta = 0$, or $p = 0$. The condition on the J's means

$$H = -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\right) \qquad \Delta = -\frac{1}{2}$$

In this case the conjecture above is a theorem [Yang, Fendley, '04; Veneziano and Wosiek, '06; Razumov, Stroganov and Z-J, '07].

Set $L = 2n+1$. Call $\Psi \in \mathcal{H}$ the ground state eigenvector. Normalize it so that its entries are *polynomials* in ζ with integer coefficients and no common factors.

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Theorem (Di Francesco, Z-J and Zuber, '06; Razumov, Stroganov and Z-J, '07)

Set $\zeta = 0$ (XXZ case). We consider only entries which have n or $n+1$ $-$'s depending on the parity of n (the other ones are zero).

- *All entries are positive integers.*
- $\Psi_{\underbrace{+\ldots+}_{n}\pm\underbrace{-\ldots-}_{n}} = 1$.
- $\Psi_{\pm\underbrace{+-\ldots+-}_{n}} = A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.
- $\sum_{\alpha_1,\ldots,\alpha_L} \Psi_{\alpha_1,\ldots,\alpha_L} = \left(\frac{3}{2}\right)^n \frac{2\times 5\times\cdots\times(3n-1)}{1\times 3\times\cdots\times(2n-1)} A_n$.

where $\pm = (-1)^n$.

A_n is the number of Alternating Sign Matrices of size n.

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Conjecture (Razumov and Stroganov, '09)

- *All entries of* Ψ *have* *positive* *coefficients as polynomials in* ζ.
- $\Psi_{+\cdots+} = \zeta^{\frac{n(n+1)}{2}} + \cdots + A_V(n+1)^2 \zeta^{\lfloor \frac{n}{2} \rfloor}$.

Conjecture (Bazhanov and Mangazeev, '09)

The norm of Ψ *is given by*

$$|\Psi|^2 = \sum_{\alpha} \Psi^2_{\alpha_1,\ldots,\alpha_L} = (4/3)^n \zeta^{n(n+1)} s_n(\zeta^{-2}) s_{-n-1}(\zeta^{-2})$$

where the $s_n(z)$ *are determined by* $s_0 = s_1 = 1$ *and the recurrence*

$$2z(z-1)(9z-1)^2(s_n s_n'' - s_n'^2) + 2(3z-1)^2(9z-1)s_n s_n'$$
$$+8(2n+1)^2 s_{n+1} s_{n-1} - [4(3n+1)(3n+2) + (9z-1)n(5n+3)] s_n^2 = 0$$

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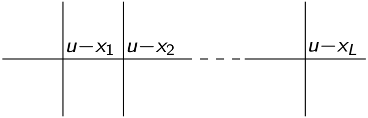
Inhomogeneous transfer matrix

Define a more general transfer matrix:

$$T(u; x_1, \ldots, x_L) =$$



where the weight at vertex i is a function of $u - x_i$.

Conjecture (Razumov and Stroganov, '09)

If $\eta = \pi/3$ and L is odd, then the inhomogeneous transfer matrix $T(u; x_1, \ldots, x_L)$ has the eigenvalue

$$\prod_{i=1}^{L} (a(u - x_i) + b(u - x_i))$$

NB: in the XXZ case, this is a theorem [R, S, and Z-J, '07]; see also [DF and Z-J, '05].

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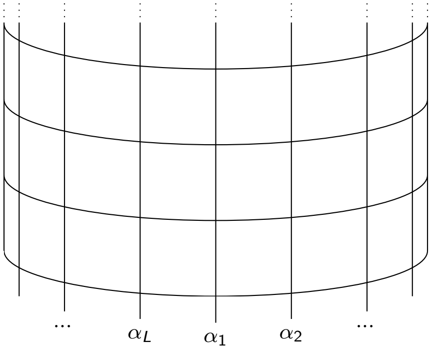
Eigenvector

Consider the eigenvector $\Psi_L(x_1,\ldots,x_L)$ of $T(u;x_1,\ldots,x_L)$ with the eigenvalue $\prod_{i=1}^{L}\left(a(u-x_i)+b(u-x_i)\right)$.

$$\Psi_{\alpha_1,\ldots,\alpha_L} =$$



Relation to quantum Knizhnik–Zamolodchikov

- $\Psi(x_1,\ldots,x_L)$ satisfies the exchange relation

$$\check{R}_i(x_{i+1}-x_i)\Psi(x_1,\ldots,x_L)=\Psi(x_1,\ldots,x_{i+1},x_i,\ldots,x_L)$$

where $\check{R}_i(x)=\frac{1}{a(x)+b(x)}\begin{pmatrix} a(x) & 0 & 0 & d(x) \\ 0 & c(x) & b(x) & 0 \\ 0 & b(x) & c(x) & 0 \\ d(x) & 0 & 0 & a(x) \end{pmatrix}_{i,i+1}$;

- Cyclic invariance

$$\Psi_{\alpha_1,\ldots,\alpha_L}(x_1,\ldots,x_L)=\Psi_{\alpha_2,\ldots,\alpha_L,\alpha_1}(x_2,\ldots,x_L,x_1)$$

These can be considered as a special case of the level 1 quantum Knizhnik–Zamolodchikov(–Bernard) equation at a cubic root of unity.

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Six-vertex case

We assume L odd, $\Delta = -\frac{1}{2} = \frac{1}{2}(q + q^{-1})$, $q = e^{2\pi i/3}$. The normalization of Ψ_L can be chosen so that its entries are polynomials (of degree $n-1$) in each variable.

Theorem (R, S and Z-J, '07)

If $\Phi(x) = (\Phi_+(x), \Phi_-(x))$ is the type I vertex operator associated to level 1 highest weight modules of $U_q(\widehat{sl(2)})$, then $\Psi(x_1, \ldots, x_L)$ coincides (up to normalization) with $\langle 0|\,\Phi(x_1)\cdots\Phi(x_L)\,|1\rangle$.
Explicitly, if $\{i : \alpha_i = +\} = \{a_1 < \cdots < a_n\}$, and $z_k = e^{2ix_k}$,

$$\Psi_{a_1,\ldots,a_n}(x_1,\ldots,x_L) = \prod_{1\le i<j\le L} (q\, z_i - q^{-1} z_j)$$

$$\oint \prod_{\ell=1}^{n} \frac{dw_\ell}{2\pi i} \prod_{\ell=1}^{n} x_{a_\ell} \frac{\prod_{\ell=1}^{n} w_\ell \prod_{1\le \ell<m\le n}[(w_m - w_\ell)(q\, w_\ell - q^{-1} w_m)]}{\prod_{\ell=1}^{n}\left[\prod_{1\le i\le a_\ell}(w_\ell - z_i)\prod_{a_\ell\le i\le L}(q\, w_\ell - q^{-1} z_i)\right]}$$

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Six-vertex cont'd: linear sum rule

For example, one can compute

$$\sum_{\alpha_1,\ldots,\alpha_L} \Psi_{\alpha_1,\ldots,\alpha_L}(x_1,\ldots,x_L) = s_{(n-1,n-1,\ldots,1,1)}(z_1,\ldots,z_L) \qquad z_k = e^{2ix_k}$$

This is closely related to the enumeration of ASMs. In fact, in the six-vertex case, there is an analogue problem in even size (twisted XXZ chain). There again,

$$\sum_{\alpha_1,\ldots,\alpha_{2n}} \Psi_{\alpha_1,\ldots,\alpha_{2n}}(x_1,\ldots,x_{2n}) = s_{(n-1,n-1,\ldots,1,1)}(z_1,\ldots,z_{2n})$$

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Eight-vertex case

Go back to the general 8-vertex case:

Conjecture

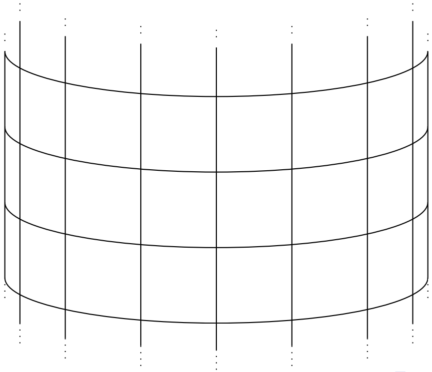
The normalization of Ψ can be chosen so that its entries are theta functions of nome p and of order $L-1$ in each variable.

(theta functions = holomorphic, doubly pseudo-periodic functions)

Definition

Define

$$Z_L(x_1,\ldots,x_L)=\sum_{\alpha_1,\ldots,\alpha_L}\Psi_{\alpha_1,\ldots,\alpha_L}(x_1,\ldots,x_L)\Psi_{\alpha_1,\ldots,\alpha_L}(-x_1,\ldots,-x_L)$$

$Z_L =$ 

Characterization

Assuming the conjecture above, we have the

Theorem

Z_L is a symmetric function of its arguments, and a theta function of nome $\sqrt{p}$ and of order $L-1$ in each.

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Half-specialization

Define

$$Z_L(0, x_1, -x_1, \ldots, x_n, -x_n) = \prod_{i=1}^{n} \vartheta^2(x_i - \eta)\vartheta^2(x_i + \eta) X_n(x_1, \ldots, x_n)$$

Then X_n is an even theta function in each of its arguments, of degree $2(2n-1)$, which is entirely determined by the recurrence relations

- $X_n(\ldots, x, x+\eta) = \prod_{i=1}^{n-2} \vartheta(x-\eta-x_i)^4 \vartheta(x-\eta+x_i)^4 \varphi(x)\varphi(x+\eta) X_{n-2}(\ldots)$.
- $X_n(\ldots, \eta+\alpha) = (\mathit{stuff}) X_{n-1}(\ldots)$ for all three $\alpha \neq 0$ s.t. $2\alpha = 0$.

Theorem

- *The recurrence relations are solved by a product of Pfaffians:*

$$X_n(x_1,\dots,x_{2n}) = A_n(x_1,\dots,x_n)B_n(x_1,\dots,x_n)$$
$$A_n(x_1,\dots,x_n) \propto \mathrm{Pf}\ f(x_i,x_j), \quad B_n(\dots) = A_{n+1}(\dots,\pi/2+\eta)$$

where f is a certain skew-symmetric elliptic functions of their arguments. (for A_{2m+1} add extra row/column)

- *A_n and B_n can also be expressed as products of two determinants (similar to elliptic versions of Tsuchiya's determinant as in [Filali, '11]).*
- *In the homogeneous limit, $A_n \to \zeta^{\lfloor n^2/2 \rfloor} s_n(\zeta^{-2})$ and $B_n \to (4/3)^n \zeta^{\lfloor (n+1)^2/2 \rfloor} s_{-n-1}(\zeta^{-2})$, and the BM bilinear recurrence relations = Plücker + linear differential relations.*

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Polynomials in the homogeneous limit

If ones sends all x_i to zero, then Z_L becomes the product of 4 polynomials in ζ (the constant term corresponding to the six-vertex limit), best expressed separately depending on parity of n:

$$
\begin{aligned}
H_{2m} &= 1,\quad 1,\quad 3+\zeta^2,\quad 26+29\zeta^2+8\zeta^4+\zeta^6\\
2^{m-1}H_{2m}(J_2) &= \quad 1,\quad 7+\zeta^2,\quad 143+99\zeta^2+13\zeta^4+\zeta^6\\
H_{2m}(J_3) &= \quad 1,\quad 2+\zeta+\zeta^2,\quad 11+12\zeta+21\zeta^2+10\zeta^3+7\zeta^4+\\
H_{2m}(J_4) &= \quad 1,\quad 2-\zeta+\zeta^2,\quad 11-12\zeta+21\zeta^2-10\zeta^3+7\zeta^4-\\
2^{m-1}H_{2m}(J_2,J_3) &= \quad 1,\quad 5+2\zeta+\zeta^2,\quad 66+63\zeta+81\zeta^2+30\zeta^3+12\zeta^4+\\
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\end{aligned}
$$

Polynomials in the homogeneous limit

If ones sends all x_i to zero, then Z_L becomes the product of 4 polynomials in ζ (the constant term corresponding to the six-vertex limit), best expressed separately depending on parity of n:

$$H_{2m} = 1, \quad 1, \quad 3+\zeta^2, \qquad 26+29\zeta^2+8\zeta^4+\zeta^6$$

$1, 1, 3, 26, 646 \ldots$ is the number of VSASMs of odd size.

$$
\begin{aligned}
2^{m-1} H_{2m}(J_2) &= \quad 1, \quad 7+\zeta^2, \qquad 143+99\zeta^2+13\zeta^4+\zeta^6 \\
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\end{aligned}
$$

$1, 2, 11, 170\ldots$ is the number of CSTCPPs.

$$
\begin{aligned}
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$1, 5, 66, 2431\ldots$ is one of the factors in the number of UUASMs.

Polynomials in the homogeneous limit

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$1, 7, 143, 8398$ is ??????

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Vertex-IRF transformation

The equations above do not enable us to compute Ψ because of lack of S^z conservation. $\longrightarrow$ use vertex-IRF transformation.

Introduce (dual) vertex-IRF intertwiners:

$$t_\pm(z)^a_b = \begin{pmatrix} \vartheta_2 \\ \vartheta_3 \end{pmatrix} ((a-b)z + 2a\eta)$$

and for any sequence $(a_i)_{i=0,\ldots,L}$ such that $a_{i+1} = a_i \pm 1$,

$$\Phi_{a_0,\ldots,a_L} = \sum_{\alpha_1,\ldots,\alpha_L=\pm} \Psi_{\alpha_1,\ldots,\alpha_L} \prod_{i=1}^{L} t_{\alpha_i}(z_i)^{a_{i-1}}_{a_i}$$

$(a_i) =$

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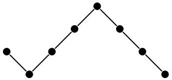
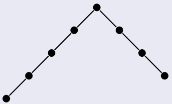
$(a_i) =$ 

The exchange relation turns into a simple recurrence formula for Φ.

Conjecture (Weston and Z-J, '11)

- $\Phi_{a_0,\ldots,a_L} = 0$ *if* $a_L \neq a_0 \pm 1$.



- *The* $\Phi_{a,\ldots,a+1}$ *are determined by the initial condition:*

$$\Phi_{a,a+1,\ldots,a+n,a+n+1,a+n,\ldots,a+1} \propto \prod_{1\le i<j\le n+1} \vartheta(z_i - z_j + 2\eta) \prod_{n+2\le i<j\le L} \vartheta(z_i - z_j + 2\eta) \prod_{i=n+2}^{L} \vartheta(z_i)\, \vartheta_{2,3}\Big(\sum_{i=1}^{n+1} z_i - \sum_{i=n+2}^{L} z_i - 2(a-n)\eta\Big)$$

and the recurrence relation: (τ_i = *permutation of* z_i *and* z_{i+1})

$$\Phi_{\ldots,b,b-1,b,\ldots} = \frac{\vartheta(2\eta\, b)\vartheta(2\eta+z_i-z_{i+1})\tau_i - \vartheta(2\eta)\vartheta(2b\,\eta+z_i-z_{i+1})}{\vartheta(2\eta(b+1))\vartheta(z_{i+1}-z_i)}\Phi_{\ldots,b,b+1,b,\ldots}$$

Prospects

- Connection to nonsymmetric elliptic Macdonad polynomials?
- Connection/appliation to the work of Rosengren?
- Connection to Painlevé VI?
- Combinatorial interpretation of all the entries of these polynomials?

IRF model:

- Can we “solve” the recurrence relations for the IRF model? (factorized expression?)
- Lashkevich and Pugai introduced elliptic vertex operators and wrote integral expressions for vacuum correlation functions which should be related to $\Phi_{a_0,\ldots,a_L}$. Use these integral expressions?
- Can we go back to the “spin” basis and prove other Bazhanov–Mangazeev conjectures?

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