Conformal invariance of double-dimer loops

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MSRI workshop on conformal invariance and statistical mechanics

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A dimer cover is a planar graph (a finite subset of $\mathbb{Z}^2$ for this talk) with a collection of disjoint edges spanning the vertices of the graph. The uniform dimer model is a random dimer model in which we simply choose uniformly at random from the set of all dimer covers.

The double dimer model is two independent copies of the dimer model superimposed. Some edges are doubled, and the edges which are not doubled are part of some loop.

Our goal is to understand the structure of these loops, in particular the long ones.

There are conjectures about the conformal invariance of these loops and their relation to CLE_4. We can't prove the relation to SLE, but we will say something about conformal invariance.

On a surface, a *finite lamination* is a collection of simply curves which are pairwise disjoint and non-peripheral (not homotopic to a puncture), modulo isotopy (meaning homotopy equivalence, loop-wise).

Let U be a planar domain, and let $U_\epsilon = U \cap \epsilon\mathbb{Z}^2$. Let $z_1, \ldots, z_k$ be points in U. For each finite lamination L in $U \setminus \{z_1, \ldots, z_k\}$, $\Pr_\epsilon(L)$ converges and depends only the conformal type of $U \setminus \{z_1, \ldots, z_k\}$.

Here we are considering only those loops in the double-dimer model which are non-contractible and non-peripheral, and asking about the resulting finite lamination L.

We consider only non-peripheral loops so that we don't have to deal with infinitely many loops surrounding a point; an alternative is to remove disks instead of points.

Single dimer Kasteleyn matrix for $\mathbb{Z}^2$: we associate to a lattice $\subset \mathbb{Z}^2$ a collection of edge weights in $\{\pm 1, \pm i\}$ (see figure), and let $K = K(w, b)$ be the matrix which is 0 when w, b are not adjacent, and the weight of the edge if w and b are adjacent. Then $|\det K|$ is the number of dimer covers of the graph (Kasteleyn's theorem).

We define

$$Kf(w) = f(w+1) - f(w-1) + if(w+i) - if(w-i),$$

which is a discrete $\bar{\partial}$ operator. So a function in the kernel of the Kastelyn matrix can be called *discrete analytic.*

To extend Kasteleyn's theorem to the double dimer model, we just need to square the determinant of K (product of partition functions).

$$\mathbb{K} = \begin{pmatrix} 0 & K \\ K^t & 0 \end{pmatrix}$$

Then $\det \mathbb{K}$ is the number of double-dimer covers.

Let $\Omega(G)$ be the set of double-dimer configurations (sets of loops and doubled edges). The number of ways a configuration with k loops may arise from two single dimer models is 2^k, so we can think of the double dimer model as a non-uniform measure on $\Omega(G)$.

We now introduce quaternionic edge weights $q = a_0 + a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $q^* = a_0 - a_1\hat{\imath} - a_2\hat{\jmath} - a_3\hat{k}$. The weight of $\omega \in \Omega$ is defined to be $\prod_{\text{cycles}}(m + m^*)$, where m is the product of the edge weights around the cycle. To be more precise, we associate each edge with a natural direction (black to white, say), and as we go around the cycle, we multiply by q if we go in the same direction as the arrow and by q^* if we're going in the opposite direction. Note that this does not depend on the vertex at which we start, since $(q_1 q_2)^* = q_2^* q_1^*$. Doubled edges count as qq^*.

Theorem 1.

$$\text{Qdet } \mathbb{K} = \sum_{\Omega} \prod_{\text{cycles}} (m + m^*).$$

Here Qdet is the quaternionic determinant (Dyson):

$$\text{Qdet } M = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{\text{cycles c of } \sigma\text{,sorted and in order}} m(c)$$

Note that the order matters here, so we have to choose some canonical way of ordering cycles (the cycle involving 1, the cycle involving the 2 (unless 2 is in 1's cycle), and so on).

Theorem 2.

$$\text{Qdet}\mathbb{K}_{n\times n} = \sqrt{\det \tilde{\mathbb{K}}_{2n\times 2n}},$$

where $\tilde{\mathbb{K}}_{2n\times 2n}$ is obtained by replacing each quaternion with a 2×2 complex matrix representing it.

Example 1. Put weights q^x on north-going edges, where x is the x-coordinate in lattice units.

Then $Z(q) = \det \mathbb{K}(q)$ counts loops with weight $q^{\text{Area}} + (q^*)^{\text{Area}}$. In particular $Z(e^{i\theta})$ counts loops with weight $2\cos(\theta\text{Area})$.

Example 2. Put weights q on a set of edges intersecting a particular straight line path (zipper) from an interior point z to ∞. This gives us a way to think about the topology of loops, because loops surrounding z have weight $q + q^*$. We can actually use this to compute the generating function of the number of loops surrounding z.

We can extend this to two interior points, because loops surrounding A gets weight $q_1 + q_1^*$, a loop surrounding B gets weight $q_2 + q_2^*$, and loops surrounding both get weight $q_1 q_2 + (q_1 q_2)^*$. These three expressions are algebraically independent, so we can extract from the partition function the generating function for the triple (X, Y, Z) of the number of loops surrounding A, B, and A & B.

Lemma: (based on Fock-Goncharov) By varying the q's, one can extract from $\det \mathbb{K}$ the contribution from any finite lamination.

This lemma says that, in principle, we may compute for each finite lamination and each graph, the probability that this lamination arises from our double dimer model. The question remains how well we can do these approximations, however, because we want to have something explicit enough to take $\epsilon \to 0$.

Theorem 3. $F(\mathbf{q}) = \lim_{\epsilon \to 0} Z_\epsilon(\mathbf{q})/Z_\epsilon(1)$ exists and is conformal invariant.

Proof idea. Take a path of weights $\mathbf{q}_t$ for $t \in [0,1]$, with $\mathbf{q}_0 = \mathbf{1}$. Then

$$\frac{d}{dt} \log Z_\epsilon(\mathbf{q}_t) = \frac{1}{2} \frac{d}{dt} \log \det \tilde{\mathbb{K}}(\mathbf{q}_t),$$

which can be written as a sum along the zippers of the Green's function $\tilde{\mathbb{K}}^{-1}(\mathbf{q}_t)$, and it's a discrete analytic function. The main difficulty is the convergence of discrete analytic functions to corresponding analytic functions. The key here is to work with boundary conditions which are nice enough to make this possible. □

Example. $m \times n$ annulus. We can graph the probability of having k loops around the inner annulus as a function of m/n, for each k (see figure).

Example. $q_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Then loops surrounding just A or B have weight 2 like most other loops, so Z just counts the loops surrounding both A and B:

$$Z = \sum_k C_k (2 + t^2)^k.$$

Theorem 4. $\mathbb{E}k = g(A, B)$, where g is the Dirichlet Green's function.

Extensions:

Can we do this on curved surfaces?

Is there a corresponding version for the Ising model?

Spanning tree/CRSF model - here there are some results.

Question: Is there a closed form for $F(\mathbf{q})$?

Answer: I don't think so. One approach to show convergence to CLE_4 is to compute a corresponding differential equation for the F coming from CLE_4, and show that the PDEs are the same.

Question: Are there closed-form formulas for certain special-case domains?

Answer: Possibly.

Question: What conditions on the boundary are required for this result?

Answer: “Temperleyan.”

Question: Is it possible that no dimer covers exist, even for small $\epsilon > 0$?

Answer: This is a point which was concealed by brushing over the boundary conditions (which are quite a mess), but with appropriate boundary conditions, they always exist.