Discrete holomorphicity and critical boundary fugacity for the $O(n)$ model on the honeycomb lattice

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MSRI workshop on conformal invariance and statistical mechanics
Lecture notes, 11:00 am, March 26, 2012
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Self-avoiding walk problem: How many ways are there to talk from A to B on a graph without retracing? We call the number of self-avoiding walks from the origin in the honeycomb lattice c_n. The following theorem is classical.

Theorem 1. $\log \mu = \lim_{n\to\infty} \frac{1}{n} \log c_n$ exists.

Conjecture 1. $\mu = \sqrt{2+\sqrt{2}}$, based on columb bas and renormalization.

Theorem 2. $\mu = \sqrt{2+\sqrt{2}}$, based on parafermonic observables

We define

$$F(z) = \sum_{\gamma(a\to z)} e^{i\sigma W(\gamma(a\to z))} x^{\ell} y^{\nu} n^{c},$$

where ℓ is the length of the walk plus the length of all the loops, ν is the number of contacts with the boundary, n is the weight of the closed loop, and W is the winding angle, σ a spin.

Lemma 1. For $n \in [-2, 2]$, set $n = 2\cos\theta$. Then for $\sigma = (\pi + 3\theta)/(4\pi)$, $x^{-1} = 2\cos((\pi - \theta)/4)$, we have

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0,$$

where p, q, r are the mid-edges adjacent to v.

Idea of proof. We group the configurations in which multiple mid-edges are visited by a loop into three different types (see figure). The contributions from each matching triple of configurations (one of each type) cancel out. □

How can we use this?

We define the following generating functions, for a trapezoidal domain with short base α of length $2L$ and height T.

$$A_{T,L}(x, y) = \sum_{\gamma(a\to b\in\alpha)} x^{\ell} y^{\nu} n^{c} \text{ and } B_{T,L}(x, y) = \sum_{\gamma(a\to b\in\beta)} x^{\ell} y^{\nu} n^{c},$$

and a term $E_{T,L}(x, y)$ including the other summands (the ones for which the path exists on one of the trapezoid legs). We then obtain that a particular linear combination of A, B, and E is equal to 1.

$$1 = \cos(3\pi/8)A^* + \cos(\pi/4)E^* + B^*, \text{ where } A* = A/Z \text{ etc.}$$

To prove that $x_{\text{critical}}^{-1} = \sqrt{2+\sqrt{2}}$, note that $x < x_c$ implies

$$B_T(x) < (x/x_c)^T B_T(x_c) \implies Z(x) < 2\prod_T (1 + B_t(x))^2 < \infty.$$

We want to show that $B_T(x) = \infty$ when $x \geq x_{\text{critical}}$. We consider walks touching β at least once. We get

$$A_{T+1} - A_T \leq x_c B_T B_{T+1}.$$

With $E_T = 0$, the preceding identity implies $B_T(x_c) \geq (\text{const})/T$, so $Z(x_c) \geq \sum_T B_T(x_c) = \infty$.

Recall that the $O(n)$ model is solveable: the R-model satisfies the Yang-Baxter equation when

$$x^{-1} = \sqrt{2 - \sqrt{2-n}},$$

which suggests what Smirnov's result might have to do with integrability.

When $y \neq 1$, letting $n = 2\cos\theta$ gives a parafermonic relation with coefficients now depending on both y and θ (i.e, some linear combination of A, B, and E equals 1).

The proof for $y \neq 1$ follows similar lines. We form triples of configurations and consider the total contribution from each triple.

One of the constants y^* that arises in this calculation has the property that $y = y^*$ is a solution of the Reflection Equation (a boundary version of the Yang-Baxter equation).

For $y = y^*$, the term involving B in the Duminil-Copin identity vanishes. Hence B can be no longer be bounded by this identity (corresponding to adsorption of the SAW on the boundary).

If we take the limit as $L \to \infty$,

$$1 = c_\alpha A_T(x_c, y) + \frac{y^* - y}{y(y^* - 1)} B_T(x_c, y),$$

which implies that $B(x_c, y) = \frac{y(y^*-1)}{y^*-y}(1 - c_\alpha A(x_c))$

We then consider the cases $1 - c_\alpha A(x_c) > 0$ and $1 - c_\alpha A(x_c) = 0$ separately. In each of these cases, we see that B diverges and thus is the dominant term.

Now recall that the Duminil-Copin identity is proved using vanishing contributions from sets of three configurations. Let us relax the constraint on x which forces these contributions to vanish. Let

$$(p - v)F(p) + \ldots = (1 - x/x_c)F(v).$$

Let $\tilde{F}_\gamma(x) = e^{i\tilde{\sigma}W(\gamma)} x^{|\gamma|} n^{c\gamma}$.

Summing over all vertices of a domain Ω one obtains, with $\tilde{\sigma} = 1 - \sigma$:

$$\sum_{\gamma: a \to \partial\Omega} \tilde{F}_\gamma(x) + (1 - x/x_c) \sum_{\gamma: a \to \Omega \setminus \partial\Omega} \tilde{F}_\gamma(x) = Z_\Omega(x)$$

Let $P(\theta, \ell)$ be the probability density function for winding angles of walks of length ℓ. Then

$$\sum_{\theta} e^{i\tilde{\sigma}\theta} P(\theta, \ell) \sim (\text{const}) \ell^{\gamma_{11} - \gamma_1 + 1},$$

where γ_{11}, γ_1 are conjectured scaling exponents corresponding to walks starting at the surface and ending in the bulk and starting at the surface and ending at the surface.

Sketch of proof. Define $G_\theta, \Omega(x)$ to be the sum over only walks with winding angle θ. We define $H_\Omega(x)$ as the sum over walks ending on the boundary. The off-critical identity can then be written more concisely. We assume the existence of γ_1 and γ_{11}, and then just substituting gives

$$\frac{\sum_\theta e^{i\tilde{\sigma}\theta} G^*_\theta(x)}{\sum_\theta G^*_\theta(x)} \sim (\text{const})(1 - x/x_c)^{-\gamma_{11} + \gamma_1 - 1}.$$

□

Conjecture 2. (from Duplantier and Saleur, using CFT heuristics) $\sum_\theta e^{i\tilde{\sigma}\theta} P(\theta, \ell) \sim l^{-\omega}$, with $\omega = \nu\kappa\tilde{\sigma}/2$ and κ is that in SLE$_\kappa$. Hence

$$-\gamma_{11} + \gamma_1 - 1 = \frac{9(2 - \kappa)^2}{8\kappa(4 - \kappa)}.$$

This agrees with independent predictions.