Boundary and bulk local operators in conformal field theory and SLE

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Notes taken by Samuel S Watson

This work was motivated by the work of Smirnov et al on discrete holomorphicity. He looked at the $O(n)$ model on a honeycomb lattice. One considers curves along edges of the lattice from a boundary point to an interior point. One obtains a measure $\mu^{\text{lattice}}(z, z_0)$ by considering Boltzmann weights coming from the $O(n)$ model. We may integrate against that measure

$$\int e^{-is\theta^{\gamma}_{z_0 \to z}} \, d\mu^{\text{lattice}}(z, z_0) = f_s(z, z_0),$$

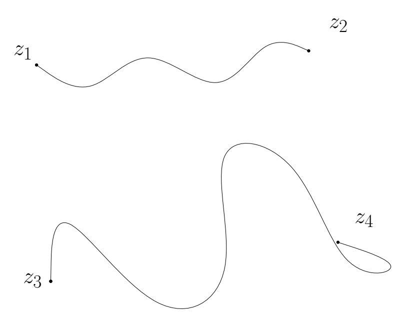
where θ denotes the winding angle. If $s = b = (6-\kappa)/(2\kappa)$ and κ has the usual relationship with n, then one obtains some cancellation for these paths, at the discrete level. Moreover, in the special case $\kappa = 3$, Smirnov was able to show that this function converges to a holomorphic function as the mesh size goes to 0. Along with some other ingredients, this is enough to show convergence to SLE_3 of the discrete path.

One may consider two interior points z_1, z_2, in which case we would expect $f(z_1, z_2)$ to depend holomorphically on both z_1 and z_2, and in particular

$$f(z_1, z_2) \sim \frac{1}{(z_1 - z_2)^{2b}}.$$

With four interior points (and in the whole plane), we would expect to a four-point function

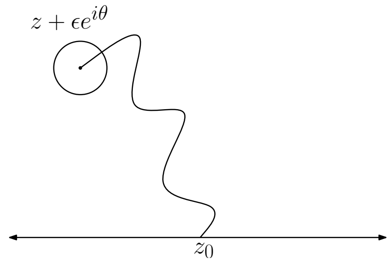
$$f(z_1, z_2, z_3, z_4) \sim [\qquad]^{4b} \, F\left(\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right)$$



We can simplify this problem somewhat by including a boundary and arguing that without loss of generality we can put two of the points on the boundary. This makes SLE tools available.

Remark: It is difficult to show (and quite possibly isn't true) that the discrete observables analogous to these four point functions satisfy the PDEs analogous to those satisfied by the continuous four-point function.

CFT point of view (2003 Bauer and Bernard - CFT of radial SLE)



We obtain (using the unnormalized version of the SLE measure, as in Greg Lawler's talk),

$$f_s(z + \epsilon e^{i\theta}, z_0) = \int e^{-is\theta^{\gamma}_{z_0 \to z}} \, d\mu(z + \epsilon e^{i\theta}, z_0),$$

and $f_z(\theta + 2\pi) = e^{is\theta} f_s(\theta)$.

Results (not theorems since they depend on CFT ideas)

1. If $s = b = (6 - \kappa)/(2\kappa)$, the limit as $\epsilon \to 0$ of f_s exists.
2. $\partial_z f_s(z, \dots) = 0$
3. $f_s(z_1, \dots, z_n)$ satisfies a complexified version of the boundary PDE.

Let's briefly remind the audience of the CFT approach. Consider usual SLE from 0 to ∞, conditioned on any even A (for example, that the path misses certain regions in the upper half plane). Then

$$\mu(z_0 | A) = \langle \phi(z_0) | A \rangle$$

$$f_0 \langle \phi(z_0) \text{anything} \rangle = \langle f \circ \phi(z_0) \int T_{\mu\nu} \alpha_\mu dn_\mu \text{anything} \rangle,$$

where the integral of the stress tensor is

$$\frac{1}{2\pi i} \int_\Gamma \alpha(z) T(z) \, dz - \frac{1}{2\pi i} \int_\Gamma \overline{\alpha}(z) \overline{T}(z) \, dz.$$

In the case where we have restriction (i.e., $\kappa = 8/3$), we are able to say what T and $\overline{T}$ should be.

We can take α according to the Loewner transform.

$$\alpha(z) = \frac{2 \, dt}{z - z_0} - \sqrt{\kappa} \, dB_t$$

Having chordal SLE, from the conformal field theory point of view, is the same as requiring that

$$\left(2L_{-2} - \frac{\kappa}{2} L_{-1}^2 \right) \phi(z_0) = 0$$

Back to the radial SLE picture, we recall the equation

$$\alpha(z) = -z \left(\frac{z + \epsilon e^{i\theta}}{z - \epsilon e^{i\theta}} \right).$$

We get

$$\left(\frac{1}{2\pi i}\int \alpha(z)T(z)\,dz + \text{c.c.}\right)\phi(\epsilon e^{i\theta}) = \left(-\frac{1}{2}\kappa\frac{\partial^2}{\partial\theta^2} + \tilde{b}\right)\phi(\epsilon e^{\theta}),$$

where $\tilde{b} = (\kappa - 4)/2$. This in turn equals

$$L_0 + \overline{L}_0 + \left(2\sum_{n=1}^{\infty}\epsilon^n e^{in\theta}L_{-n} + 2\sum_{n=1}^{\infty}\epsilon^n e^{-in\theta}\overline{L}_{-n}\right)\phi(\epsilon e^{\theta}).$$

Write $\phi(\epsilon e^{i\theta}) =: \phi(\theta)$. We look for solutions of this equation of the form

$$\phi(\theta) = e^{is\theta}\sum_{n\in\mathbb{Z}} e^{im\theta}\phi_{s+m},$$

since it will have spin s. As $\epsilon \to 0$, we get

$$(L_0 + \overline{L}_0)\phi_s = \left(\frac{1}{2}\kappa s^2 + \tilde{b}\right)\phi_s.$$

If $s = b$, then the expression in parentheses equals b. We obtain

$$\begin{array}{ll} (L_0 + \overline{L}_0)\phi_b = b\phi_b & L_0\phi_b = b\phi b \\ (L_0 - \overline{L}_0)\phi_b = b\phi_b & \overline{L}_0\phi_b = 0. \end{array}$$

It requires a lot more work to show that

$$\overline{L}_{-1}\phi_b = \frac{\partial}{\partial\bar{z}}\phi_b = 0.$$

We also get

$$\left(L_{-2} - \frac{\kappa}{4}L_{-1}^2\right)\phi_b = 0.$$

which shows that these observables satisfy the appropriate complexifications of the second-order PDEs mentioned previously.

It turns out that the only way in which these equations are consistent is if $s \in \{-b, b, 0\}$.