Directed polymers and KPZ universality

Timo Seppäläinen

Department of Mathematics
University of Wisconsin-Madison

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(Incomplete) review of directed polymers in i.i.d. random environments, especially KPZ universality in 1+1 dimensions

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1.1. Weak and strong disorder.

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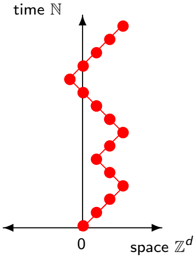
2. $1 + 1$ dimensions

2.1. KPZ universality

2.2. The three exactly solvable models.

2.3. Specific results for the log-gamma polymer: stationary process, fluctuation exponents, tropical combinatorics.

The model: directed polymer in a random environment



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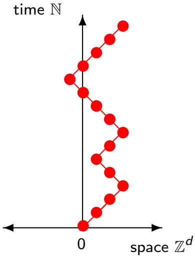

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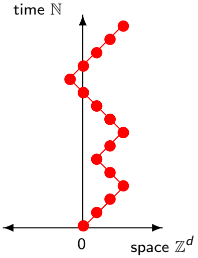

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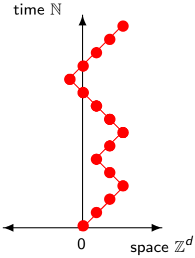

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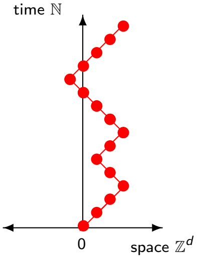

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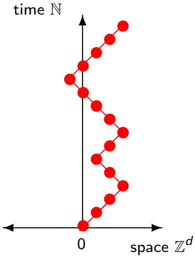

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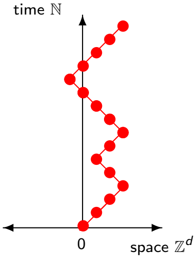

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$\mathbb{P}$ probability distribution on ω, often $\{\omega(k,x)\}$ i.i.d.

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- Dependence on β and d

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Early rigorous results: diffusive behavior for $d \geq 3$ and small $\beta > 0$:

1988 Imbrie and Spencer: $n^{-1} E^Q(|x(n)|^2) \to c \quad \mathbb{P}$-a.s.

1989 Bolthausen: quenched CLT for $n^{-1/2} x(n)$.

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Kolmogorov's 0-1 law: $\mathbb{P}(W_\infty > 0) = 0$ or 1.

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Definition. $\begin{cases} \text{Weak disorder:} & W_\infty > 0 \\ \text{Strong disorder:} & W_\infty = 0. \end{cases}$

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Theorem. $\exists \beta_c \in [0, \infty]$ such that

$$\beta \in [0, \beta_c) \implies \text{weak disorder}$$

$$\beta \in (\beta_c, \infty) \implies \text{strong disorder}$$

For $d \in \{1, 2\}$ $\beta_c = 0$, while for $d \geq 3$ $\beta_c \in (0, \infty]$.

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Open question: Are these always the same?

In $d \in \{1,2\}$ $\beta_c = \beta'_c = 0$.

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Theorem. Under $d \geq 3$ and weak disorder,

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Proof idea. Construct RWRE Q^ω using $W_\infty > 0$ as a density.

[Comets and Yoshida, 2006]

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Sufficient conditions for very strong disorder:

- $d = 1$ or 2
- $\beta\lambda'(\beta) - \lambda(\beta) > \log(2d)$. True for some distributions if β large enough.

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$g(\omega, z_{1,\ell})$ is a function on $\boldsymbol{\Omega}_\ell = \Omega \times \mathcal{R}^\ell$.

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Kernel p on $\mathbf{\Omega}_\ell$: $p(\eta, S_z\eta) = |\mathcal{R}|^{-1}$ for $\eta = (\omega, z_{1,\ell})$.

Entropy

For $\mu \in \mathcal{M}_1(\mathbf{\Omega}_\ell)$, q Markov kernel on $\mathbf{\Omega}_\ell$, usual relative entropy on $\mathbf{\Omega}_\ell^2$:

$$H(\mu \times q \,|\, \mu \times p) \;=\; \int_{\mathbf{\Omega}_\ell} \sum_{z \in \mathcal{R}} q(\eta, S_z\eta) \log \frac{q(\eta, S_z\eta)}{p(\eta, S_z\eta)}\, \mu(d\eta).$$

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The effect of $\mathbb{P}$ in the background?

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$H_{\mathbb{P}}$ convex but not lower semicontinuous.

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- Analogous result for point-to-point free energy.

Quenched weak LDP (large deviation principle) under Q_n.

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IID environment, directed walk $\quad\Rightarrow\quad$ full LDP holds.

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2.1. KPZ and EW universality

Two different universality classes for 1+1 dim systems.

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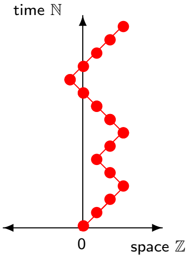
Edwards-Wilkinson (EW)

- time $\sim n$, spatial correlations $\sim n^{1/2}$, fluctuations $\sim n^{1/4}$
- Gaussian limits

KPZ class: 1+1 dim directed polymer



$\{\omega(k,x)\}$ i.i.d. under $\mathbb{P}$

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In KPZ class also

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- zero-temperature polymer, or last-passage percolation model
- Other 1+1 dim growth models (PNG, ballistic deposition)
- particle systems with drift and nonlinear flux function (ASEP, ZRP)

EW class: Random average process (RAP)

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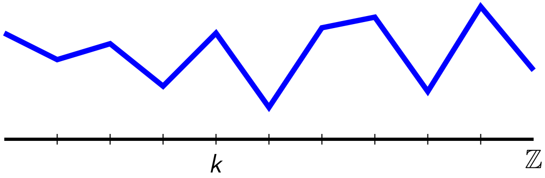

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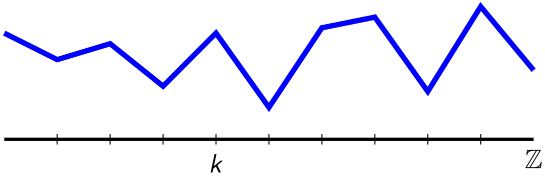

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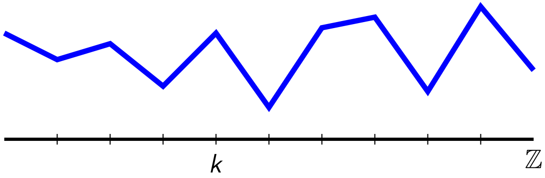

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$\omega_{t,k} = (\omega_{t,k}(j) : |\,j\,| \le R)$ random probability vectors, IID over (t, k)

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Scaled height process

$$z_n(t,r) = n^{-1/4}\big\{\sigma_{\lfloor nt \rfloor}(-\lfloor ntv \rfloor + \lfloor r\sqrt{n}\,\rfloor) - \mu_0 r\sqrt{n}\,\big\}, \quad (t,r) \in \mathbb{R}_+ \times \mathbb{R}.$$

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Theorem. [Balázs, Rassoul-Agha, S. 2006] $z_n(t,r) \Rightarrow Z(t,r)$ where Z is the Gaussian process

$$Z(t,r) \;=\; c_1 \iint\limits_{[0,t]\times\mathbb{R}} \varphi_{\sigma^2(t-s)}(r-x)\,dW(s,x) \;+\; c_2 \int\limits_{\mathbb{R}} \varphi_{\sigma^2 t}(r-x)B(x)\,dx$$

Edwards-Wilkinson (EW) universality

RAP is an example from the **EW universality class**.

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RAP is an example from the **EW universality class**.

In this class also

- current of independent random walks (incl. RWRE)
- symmetric simple exclusion process
- Hammersley's serial harness process

KPZ equation

1986 Kardar, Parisi and Zhang: general model for height function $h(t,x)$ of a 1+1 dimensional growing interface:

$$h_t \;=\; \tfrac{1}{2}\, h_{xx} \,+\, \tfrac{1}{2}\,(h_x)^2 \,+\, \dot{W}$$

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Define $h \;=\; \log Z$ as the **Hopf-Cole solution** of KPZ.

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  Cross-over distribution because it has

$$\begin{cases} \text{Tracy-Widom } F_{\text{GUE}} \text{ limit in the scale } t^{1/3} & \text{as } t \nearrow \infty \\ \text{Gaussian limit in the scale } t^{1/4} & \text{as } t \searrow 0. \end{cases}$$

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- Universal cross-over between KPZ class and EW class.
- Limit of discrete models when asymmetry or noise suitably tuned to zero as the limit is taken.
- First result **Bertini and Giacomin 1997:** height function of weakly asymmetric simple exclusion process converges to Hopf-Cole solution of KPZ.

2. 1+1 dim systems
2.2 Exactly solvable directed polymers

Three exactly solvable 1+1 dim models (positive temperature)

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Next brief look at the two discrete models.

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$$+ B_3(s_3) - B_3(s_2) + \cdots + B_n(t) - B_n(s_{n-1})\big)\Big]\, ds_{1,n-1}$$

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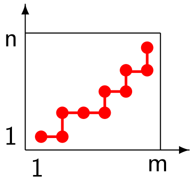
Log-gamma polymer



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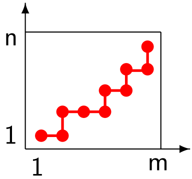

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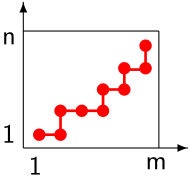

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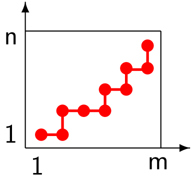

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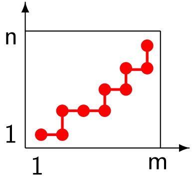

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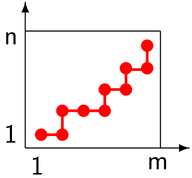

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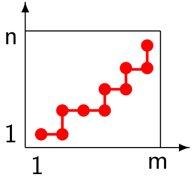

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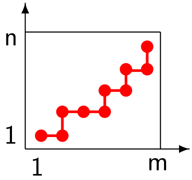

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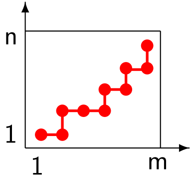

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2. It can be "solved" with ideas from tropical combinatorics

This yields

- an explicit formula for the Laplace transform of Z

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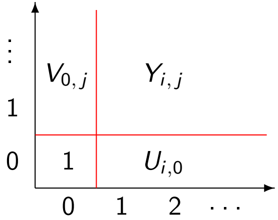

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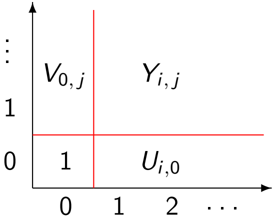

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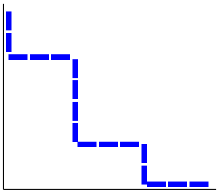

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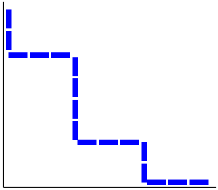

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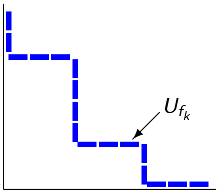

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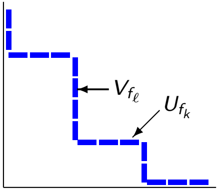

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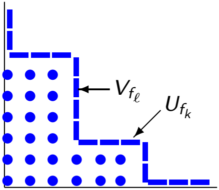

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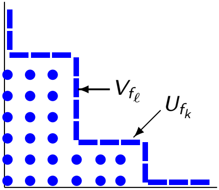

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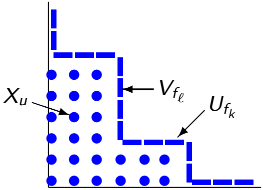

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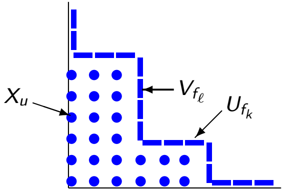

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V_{f_ℓ}
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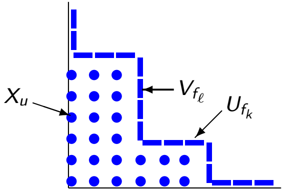

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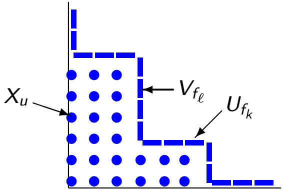

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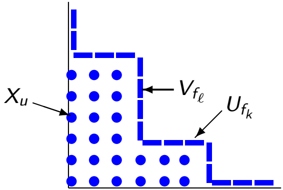
There is an analogous property for last-passage percolation with exponential weights that is a generalization of Burke's Theorem (Output Theorem) for M/M/1 queues.



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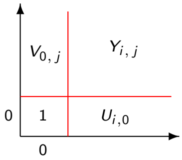
There is an analogous property for last-passage percolation with exponential weights that is a generalization of Burke's Theorem (Output Theorem) for M/M/1 queues.

Hence we could call this the "Burke property" of the log-gamma polymer.

Taking advantage of the stationarity



Initial weights ($i, j \in \mathbb{N}$):

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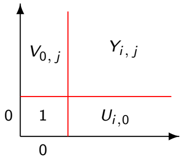

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Coupling of two log-gamma models:

- Original one with IID bulk weights, paths $(1, 1) \to (m, n)$
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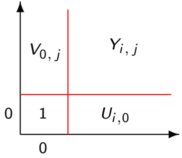

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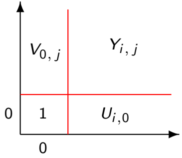

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Let us look at fluctuation exponents for $\log Z$.

Fluctuation exponents: stationary case

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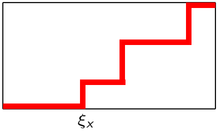

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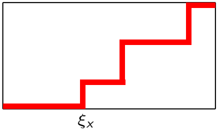

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$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

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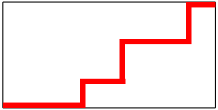

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Theorem. For the stationary case,

$$\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big]$$

Remark: polygamma functions

$$\Psi_n(s) = \frac{d^{n+1}}{ds^{n+1}} \log \Gamma(s), \qquad n \geq 0$$

These appear naturally because for $Y^{-1} \sim \text{Gamma}(\mu)$

$$\mathbb{E}(\log Y) = -\Psi_0(\mu) \qquad \text{(digamma function)}$$

$$\mathbb{V}\text{ar}(\log Y) = \Psi_1(\mu) \qquad \text{(trigamma function)}$$

Fluctuation exponent: stationary case

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\, m - N\Psi_1(\mu - \theta)\,| \leq CN^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \leq CN^{2/3} \qquad (1)$$

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For (m, n) as in (1), $\quad C_1 N^{2/3} \le \mathbb{V}\mathrm{ar}(\log Z_{m,n}) \le C_2 N^{2/3}$.

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Theorem: Off-characteristic CLT

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$.
Then

$$N^{-\alpha/2}\Big\{\log Z_{m,n} - \mathbb{E}\big(\log Z_{m,n}\big)\Big\} \Rightarrow \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$

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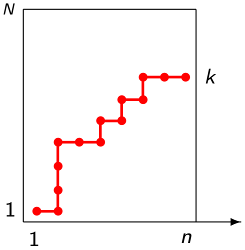
Remark. Corresponding bounds exist for path with KPZ exponent $2/3$.

Combinatorial approach to log-gamma polymer



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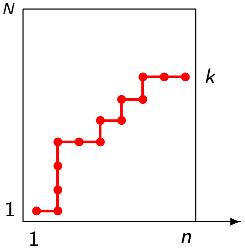

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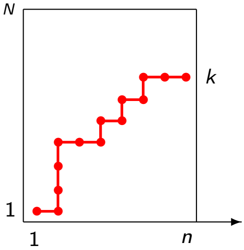

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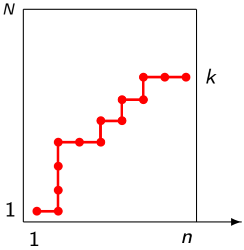

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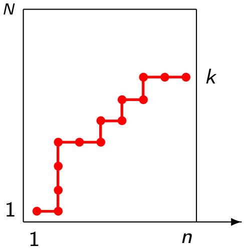

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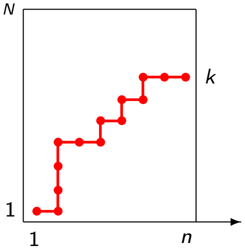

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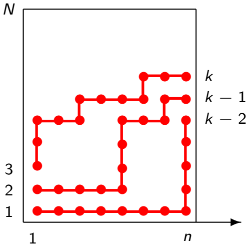

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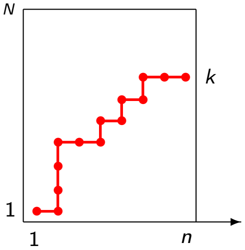

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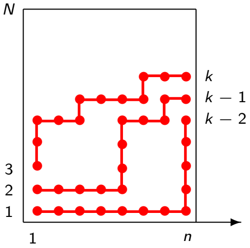

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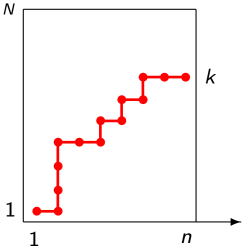

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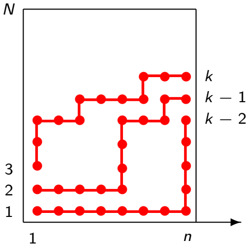

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		$z_{22}(n)$		$z_{21}(n)$		*polymer*
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- Details not illuminating.

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Theory of **Markov functions** is useful here.

General Markov functions idea

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$$\begin{array}{c} T \\ \Pi \big\downarrow \\ T \end{array}$$

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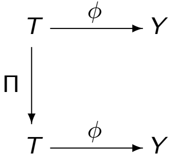

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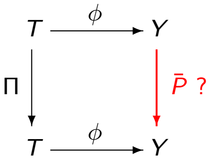

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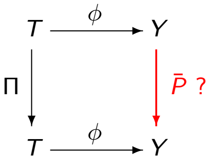

Sufficient condition. Suppose $\exists$ (positive but not necessary stochastic) kernels $P : Y \to Y$ and $K : Y \to T$ such that

$$K(y, \phi^{-1}(y)) = 1 \quad \text{and} \quad K \circ \Pi = P \circ K$$

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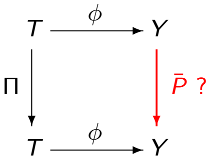

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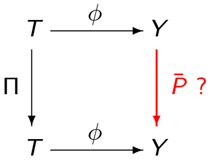

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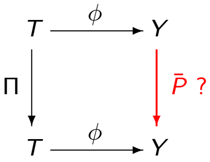

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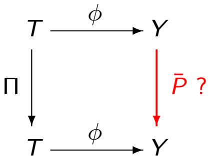

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Markov functions idea, continued

Then $\bar{K} \circ \Pi = \bar{P} \circ \bar{K}$

$$\begin{array}{ccc} T & \overset{\phi}{\underset{\bar{K}}{\rightleftarrows}} & Y \\ \Pi \downarrow & & \downarrow \bar{P} \\ T & \overset{\phi}{\underset{\bar{K}}{\rightleftarrows}} & Y \end{array}$$

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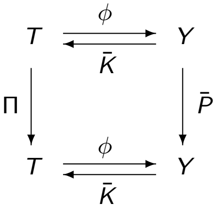

Theorem. [Rogers and Pitman, 1981]

If $z(n)$ starts with distribution $\bar{K}(y, dz)$, then $y(n)$ is Markov in its own filtration with transition $\bar{P}$ and initial state $y(0) = y$.

Application of Markov functions to our array $z(n)$

Spaces: $\mathbb{T}_N$ = space of arrays z of size N

$\mathbb{Y}_N = (0, \infty)^N$ = space of positive N-vectors y.

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and intertwining kernel $K : \mathbb{Y}_N \to \mathbb{T}_N$ by

$$K(y, dz) = \prod_{1\le\ell\le k<N} \left(\frac{z_{k,\ell}}{z_{k+1,\ell}}\right)^{\theta_{k+1}-\theta_\ell}$$

$$\times \ \exp\left(-\frac{z_{k,\ell}}{z_{k+1,\ell}} - \frac{z_{k+1,\ell+1}}{z_{k,\ell}}\right) \frac{dz_{k,\ell}}{z_{k,\ell}} \prod_{\ell=1}^{N} \delta_{y_\ell}(dz_{N,\ell})$$

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Beneficial because known special functions diagonalize the transition kernel.

Whittaker functions

$GL(N,\mathbb{R})$-Whittaker function is given for $y \in \mathbb{Y}_N$, with $\lambda \in \mathbb{C}^N$, by

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Intertwining develops into

$$\int_{(0,\infty)^N} \frac{\Psi_{\theta+\lambda}(\tilde{y})}{\Psi_\theta(\tilde{y})} \bar{P}_n(y, d\tilde{y}) \ = \ \left(\prod_{i=1}^N \frac{\Gamma(\gamma_{n,i} + \lambda_i)}{\Gamma(\gamma_{n,i})} \right) \frac{\Psi_{\theta+\lambda}(y)}{\Psi_\theta(y)}$$

Goal: polymer partition function $z_{N,1}(n)$ (first element of bottom row).

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Utilizing Whittaker functions (analogous to Fourier analysis) find

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Future goal: asymptotics for distribution of $\log z_{N,1}(n)$?

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Theorem. These exponents valid for stationary semidiscrete polymer. Upper bounds valid for model without boundaries. [Moreno, S, Valkó]

Explicit large deviations for $\log Z$

L.m.g.f. of $\log Y$, $Y \sim \Gamma^{-1}(\mu)$:

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Theorem. [Georgiou, S 2011]

$$\Lambda_{s,t}(\xi) = \begin{cases} p(s,t)\xi & \xi < 0 \\ \inf_{\theta \in (\xi,\mu)} \big\{ tM_\theta(\xi) \; - \; sM_{\mu-\theta}(-\xi) \big\} & 0 \le \xi < \mu \\ \infty & \xi \ge \mu. \end{cases}$$

- $\Lambda_{s,t}$ linear on $\mathbb{R}_-$ because for $r < p(s,t)$

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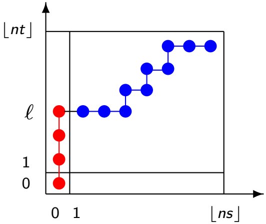
- Proof of formula for $\Lambda_{s,t}$ goes by first finding $J_{s,t}$ and then convex conjugation.

Starting point for proof of large deviations



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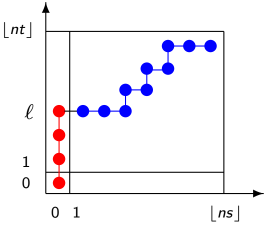

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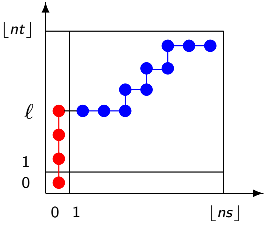

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Now we know LDP for log(l.h.s) and can extract log Z from the r.h.s.