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- Title: Random walks on finite graphs and random interlacements
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1. Consider a simple random walk (X_n) on the torus $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$. Facts about the complement of the trajectory $\mathcal{V}^t = \mathbb{T}_N^d \setminus \{X_1, \ldots, X_t\}$:

   - $\mathcal{V}^{C_d(\log N)N^d} = \emptyset$ with high probability as $N \to \infty$.
   - $\frac{\mathcal{V}^{uN^d}}{N^d} \to \rho_u \in (0,1)$.

   Let $\mathcal{C}^t$ and $\mathcal{C}_2^t$ be the largest and the second largest components of $\mathcal{V}^t$.

   - (Benjamini-Sznitman '06)$d \geq d_0, u \leq \bar{u}_d$,
   $$P(|\mathcal{C}^{uN^d}| > \epsilon_{u,d}N^d) \to 1 \text{ as } N \to \infty. \qquad (*)$$
   - (T.-Windisch'11)
   $d \geq 3, u \leq \bar{u}_d$, $(*)$ still holds.
   $d \geq 3, u > \bar{\bar{u}}_d$, $P(|\mathcal{C}^{uN^d}| \geq \log^{\lambda(u,d)} N) \to 0$.

   Local picture of the random walk:

   - (Windisch '08) For $A \subset \mathbb{Z}^d$, let $w_{A,u}^N$ be the law of $\mathcal{V}^{uN^d} \cap A$ on $\{0,1\}^A$. Then $\omega_{A,u}^N$ converges to a law $w_{A,u}$.

   We will describe the law $w_{A,u}$.

2. Random interlacement

   - equilibrium measure $e_A(x) := 1_A(x)P_x(\text{never return to} A)$;
   - For each x, there are $J_x \sim Poisson(u \cdot e_A(x))$ independent random walks starting from x.
   - $\mathcal{V}_A^u$: complement of their ranges.

   Claim: $w_{A,u}$ is the law of $\mathcal{V}_A^u$.
   Theorem(Sznitman '07) $(w_{A,u})$ is consistent, i.e., if $A' \subset A$, then the restriction of $w_{A,u}$ to A' is $w_{A',u}$. $\Longrightarrow$existence of a law w_u on $\{0,1\}^{\mathbb{Z}^d}$.

   Reasons to analyse w_u:

   1) w_u is mixing on $\mathbb{Z}^d$;
   2) it inherits independent structure from the Poisson point process.

   Observations:

- $P(o \in \mathcal{V}^u) = \exp(-u \operatorname{Cap}(o)) = \rho_u$.
- $P([0,L]^d \subset \mathcal{V}^u) = \exp(-u \operatorname{Cap}([0,L]^d))$
- $P(x \in \mathcal{V}^u, y \in \mathcal{V}^u) - P(x \in \mathcal{V}^u) \le \frac{C_d}{|x-y|^{d-2}}$.

Lemma. For disjoint $K_1, K_2 \subset \mathbb{Z}^d$ and functions $f_1, f_2 \in [0,1]$ on K_1, K_2,

$$\operatorname{Cov}(f_1, f_2) \le c_d u \frac{\operatorname{Cap}(K_1)\operatorname{Cap}(K_2)}{\operatorname{dist}(K_1,K_2)^{d-2}}$$

Theorem(Sznitman '07)
If $d \ge 7, u < \bar{u}_d$, then $P(o \leftrightarrow \infty) > 0$.
If $d \ge 3, u \ge \bar{\bar{u}}_d$, then $P(o \leftrightarrow \infty) = 0$.
Theorem(Sidoravicius-Sznitman '09)
If $d \ge 3, u < \bar{u}_d$, then $P(o \leftrightarrow \infty) > 0$.
If $d \ge 3, u \ge \bar{\bar{u}}_d$, then $P(|\mathcal{C}_o^u| > v) \le \exp(-c_{u,d} v^\delta)$, where $\mathcal{C}_o^u \subset \mathcal{V}^u$ denotes the component that contains o.
Theorem(T. '10)
If $d \ge 5, u \le \bar{\bar{u}}_d$, then $P(N < |\mathcal{C}_o^u| < \infty) \le \exp(-C_{u,d} N^\delta)$.

3. The critical value
Define $\eta(u) = P(0 \leftrightarrow \infty)$.
$u_* := \inf\{u \ge 0, \eta(u) = 0\}$.
Theorem(T.-Windisch '11)
For fixed $u, \alpha > 0$, $\epsilon \in (0,1)$ and a closed ball A, there exists a coupling such that

$$P(\mathcal{V}^{u(1+\epsilon)} \cap A \subset \mathcal{V}_{N,A}^{uN^d} \subset \mathcal{V}^{u(1-\epsilon)} \cap A) \ge 1 - N^{-\alpha}.$$

Theorem(T. '08)
For a transient graph $G = (V, E)$, $\exists$random subset $V^u \subset V$ such that

$$P(K \subset V^u) = \exp(-u \operatorname{Cap}(K)).$$

Theorem(T., Tykesson '08, '12)
If the graph G is not amenable, then $u_* \in (0, \infty)$.
Theorem(T. '08) FKG inequality on $\mathcal{V}^u$.
Theorem(T., Windisch '11)
If $d \ge 5, u < \bar{u}_d$, then $P(|\mathcal{C}_2^{uN^d}| > \log^{\lambda(u,d)} N) \to 0$ and $|\mathcal{C}^{uN^d}|/N^d \to \eta(u)$, as $N \to \infty$.
If $d \ge 3, u > u_*$, then $P(|\mathcal{C}^{uN^d}| > \epsilon N^d) \to 0$ for all $\epsilon > 0$.
Theorem(T. '08)
For a d-regular tree, $u_* = \frac{d(d-1)\log(d-1)}{(d-2)^2}$.

Now consider a sequence (G_N) of d-regular graphs (G_N has N vertices) that satisfy:

1) The smallest circle on $G_N \geq \epsilon \log N$.
2) Spectral gap of G_N is $\geq c > 0$, uniformly in N.
Theorem(Čený-T.-Windisch '11) Assume (1), (2).
If $u < u_*$, then $P(|\mathcal{C}^{uN}| > \epsilon(u,d)N) \to 1$.
If $u > u_*$, then $P(|\mathcal{C}^{uN}| > k_{\epsilon,u} \log N) \to 0$.

Remarks: Condition 1) can be replaced by
1') $\forall x$, the ball $B(x, \epsilon \log N)$ contains at most one cycle.

4. Random regular graphs
   Fix $n \geq d$ such that nd is even. Let $\mathcal{G}_n^d$ be the set of d-regular graphs with n vertices. Let G be a graph picked randomly from $\mathcal{G}_n^d$ (Bollobas's 'pairing construction'). It is known that

   - (Cooper-Frize '11)
     Suppose G is sampled uniformly in $\mathcal{G}_n^d$. Then $\mathcal{V}^u$ is distributed as a random graph with given degree sequence $\overrightarrow{d} = (d_1, \ldots, d_{|\mathcal{V}^u|})$.
   - (Molloy, Reed, Hatami, Janson, Luczak)
     For $\overrightarrow{d}_n = (d_1, \ldots, d_n)$, let $Q(\overrightarrow{d}_n) = \frac{\sum d_i^2}{\sum d_i} - 2$.
     If $Q(\overrightarrow{d}_n) \geq c > 0$, then $\lim_{n\to\infty} P(|\mathcal{C}| > \epsilon n) = 1$.
     If $Q(\overrightarrow{d}_n) \leq c < 0$, then $\lim_{n\to\infty} P(|\mathcal{C}| > K \log n) = 0$.
     If $Q(\overrightarrow{d}_n) \leq \lambda n^{-1/3}$, then $|\mathcal{C}|/n^{2/3}$ is tight.

Using these results, we can show
Theorem (T.-Čený '12) For $u_n = u_* + \epsilon_n$, $|\epsilon_n| \leq \lambda n^{-1/3}$,

$$\frac{\mathcal{C}^{u_n n}}{n^{2/3}} \text{ is tight in } (0, \infty).$$