Introduction to Derived Categories

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Amnon Yekutieli

Department of Mathematics
Ben Gurion University

Notes available at `http://www.math.bgu.ac.il/~amyekut/lectures`

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Here is an outline of the mini-course.

The first lecture will be about the general theory of derived categories.

1. The Homotopy Category
2. The Derived Category
3. Derived Functors
4. Resolutions
5. DG Algebras (new section; change of numbering below)

The second lecture will be on more specialized topics, leaning towards noncommutative algebraic geometry.

5. Commutative Dualizing Complexes
6. Noncommutative Dualizing Complexes
7. Tilting Complexes and Derived Morita Theory
8. Rigid Dualizing Complexes

Due to the time constraint I had to leave out some important topics (such as DG algebras).

1. The Homotopy Category

Suppose M is an abelian category.

The main examples for us are these:

- A is a ring, and $\mathsf{M} = \operatorname{Mod} A$, the category of left A-modules.
- $(X, \mathcal{A})$ is a ringed space, and $\mathsf{M} = \operatorname{Mod} \mathcal{A}$, the category of sheaves of left $\mathcal{A}$-modules.

A complex in M is a diagram

$$M = \left(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \to \cdots \right)$$

in M such that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^{i} = 0$.

A homomorphism of complexes $\phi : M \to N$ is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{-1} & \xrightarrow{\mathrm{d}_M^{-1}} & M^0 & \xrightarrow{\mathrm{d}_M^{0}} & M^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi^{-1}} & & \downarrow{\scriptstyle \phi^{0}} & & \downarrow{\scriptstyle \phi^{1}} & & \\ \cdots & \longrightarrow & N^{-1} & \xrightarrow[\mathrm{d}_N^{-1}]{} & N^0 & \xrightarrow[\mathrm{d}_N^{0}]{} & N^1 & \longrightarrow & \cdots \end{array} \tag{1.1}$$

in M.

Let us denote by $\mathbf{C}(\mathsf{M})$ the category of complexes in M.

It is again an abelian category; but it is also a differential graded (DG) category, as we now explain.

Given $M, N \in \mathsf{C}(\mathsf{M})$ we let

$$\mathrm{Hom}_{\mathsf{M}}(M,N)^i := \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M^j, N^{j+i})$$

and

$$\mathrm{Hom}_{\mathsf{M}}(M,N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M,N)^i.$$

For $\phi \in \mathrm{Hom}_{\mathsf{M}}(M,N)^i$ we let

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \phi \circ \mathrm{d}_M.$$

In this way $\mathrm{Hom}_{\mathsf{M}}(M,N)$ becomes a complex of abelian groups, i.e. a DG $\mathbb{Z}$-module.

Note that the abelian structure of $\mathsf{C}(\mathsf{M})$ can be recovered from the DG structure as follows:

$$\mathrm{Hom}_{\mathsf{C}(\mathsf{M})}(M,N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathsf{M}}(M,N)\big),$$

the set of 0-cocycles.

Indeed, for $\phi : M \to N$ of degree 0 the condition $\mathrm{d}(\phi) = 0$ is equivalent to the commutativity of the diagram (1.1).

Next we define the homotopy category $\mathsf{K}(\mathsf{M})$.

Its objects are the complexes in M (same as $\mathsf{C}(\mathsf{M})$), and

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{M})}(M,N) = \mathrm{H}^0\big(\mathrm{Hom}_{\mathsf{M}}(M,N)\big).$$

In other words, these are homotopy classes of homomorphisms $\phi : M \to N$ in $\mathsf{C}(\mathsf{M})$.

There is an additive functor $\mathsf{C}(\mathsf{M}) \to \mathsf{K}(\mathsf{M})$, which is the identity on objects and surjective on morphisms.

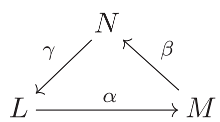
The additive category $\mathsf{K}(\mathsf{M})$ is no longer abelian – it is a triangulated category. Let me explain what this means.

Suppose K is an additive category, with an automorphism T called the translation (or shift, or suspension).

A triangle in K is a diagram of morphisms of this sort:

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L).$$

The name comes from the alternative typesetting



A triangulated category structure on K is a set of triangles called distinguished triangles, satisfying a list of axioms (that are not so important for us).

Details can be found in the references [Ye5], [Sc], [Ha], [We], [KS1], [Ne2] or [LH].

The translation T of the category $\mathbf{K}(\mathsf{M})$ is defined as follows.

On objects we take $T(M)^i := M^{i+1}$ and $\mathrm{d}_{T(M)} := -\mathrm{d}_M$. On morphisms it is $T(\phi)^i := \phi^{i+1}$.

Given a homomorphism $\alpha : L \to M$ in $\mathbf{C}(\mathsf{M})$, its cone is the complex

$$\mathrm{cone}(\alpha) := T(L) \oplus M = \begin{bmatrix} T(L) \\ M \end{bmatrix}$$

with differential (in matrix notation)

$$\mathrm{d} := \begin{bmatrix} T(\mathrm{d}_L) & 0 \\ T(\alpha) & \mathrm{d}_M \end{bmatrix}.$$

There are canonical homomorphisms $M \to \mathrm{cone}(\alpha)$ and $\mathrm{cone}(\alpha) \to T(L)$ in $\mathbf{C}(\mathsf{M})$.

A triangle in $\mathbf{K}(\mathsf{M})$ is distinguished if it is isomorphic, as a diagram in $\mathbf{K}(\mathsf{M})$, to the triangle

$$L \xrightarrow{\alpha} M \to \mathrm{cone}(\alpha) \to T(L)$$

for some homomorphism $\alpha : L \to M$ in $\mathbf{C}(\mathsf{M})$.

A calculation shows that $\mathbf{K}(\mathsf{M})$ is indeed triangulated (i.e. the axioms that I did not specify are satisfied).

For $M \in \mathbf{K}(\mathsf{M})$ and $i \in \mathbb{Z}$ we will write

$$M[i] := T^i(M),$$

the i-th translation of M.

The relation between distinguished triangles and exact sequences will be mentioned later.

Suppose K and K' are triangulated categories. A triangulated functor $F : \mathsf{K} \to \mathsf{K}'$ is an additive functor that commutes with the translations, and sends distinguished triangles to distinguished triangles.

Example 1.2. Let $F : \mathsf{M} \to \mathsf{M}'$ be an additive functor (not necessarily exact) between abelian categories.

Extend F to a functor

$$\mathbf{C}(F) : \mathbf{C}(\mathsf{M}) \to \mathbf{C}(\mathsf{M}')$$

in the obvious way, namely

$$\mathbf{C}(F)(M)^i := F(M^i)$$

for a complex $M = \{M^i\}_{i \in \mathbb{Z}}$.

The functor $\mathbf{C}(F)$ respects homotopies, so we get an additive functor

$$\mathbf{K}(F) : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{M}').$$

This is a triangulated functor.

2. The Derived Category

As before M is an abelian category.

Given a complex $M \in \mathbf{C}(\mathsf{M})$, we can consider its cohomologies

$$\mathrm{H}^i(M) := \ker(\mathrm{d}^i_M) / \operatorname{im}(\mathrm{d}^{i-1}_M) \in \mathsf{M}.$$

Since the cohomologies are homotopy-invariant, we get additive functors

$$\mathrm{H}^i : \mathbf{K}(\mathsf{M}) \to \mathsf{M}.$$

A morphism $\psi : M \to N$ in $\mathbf{K}(\mathsf{M})$ is called a quasi-isomorphism if $\mathrm{H}^i(\psi)$ are isomorphisms for all i.

Let us denote by $\mathbf{S}(\mathsf{M})$ the set of all quasi-isomorphisms in $\mathbf{K}(\mathsf{M})$.

Clearly $\mathbf{S}(\mathsf{M})$ is a multiplicatively closed set, i.e. the composition of two quasi-isomorphisms is a quasi-isomorphism.

A calculation shows that $\mathbf{S}(\mathsf{M})$ is a left and right denominator set (as in ring theory).

It follows that the Ore localization $\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$ exists. This is an additive category, with object set

$$\mathrm{Ob}(\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}) = \mathrm{Ob}(\mathbf{K}(\mathsf{M})).$$

There is a functor

$$Q : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$$

called the localization functor, which is the identity on objects.

Every morphism $\chi : M \to N$ in $\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$ can be written as

$$\chi = Q(\phi_1) \circ Q(\psi_1^{-1}) = Q(\psi_2^{-1}) \circ Q(\phi_2)$$

for some $\phi_i \in \mathbf{K}(\mathsf{M})$ and $\psi_i \in \mathbf{S}(\mathsf{M})$.

The category $\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$ inherits a triangulated structure from $\mathbf{K}(\mathsf{M})$, and the localization functor Q is triangulated.

There is a universal property: given a triangulated functor

$$F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$$

to a triangulated category E, such that $F(\psi)$ is an isomorphism for every $\psi \in \mathbf{S}(\mathsf{M})$, there exists a unique triangulated functor

$$F_{\mathbf{S}(\mathsf{M})} : \mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})} \to \mathsf{E}$$

such that

$$F_{\mathbf{S}(\mathsf{M})} \circ Q = F.$$

Definition 2.1. The derived category of the abelian category M is the triangulated category

$$\mathbf{D}(\mathsf{M}) := \mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}.$$

The derived category was introduced by Grothendieck and Verdier around 1960. The first published material is the book “Residues and Duality” [Ha] from 1966, written by Hartshorne following notes by Grothendieck.

Let $\mathbf{D}(\mathsf{M})^0$ be the full subcategory of $\mathbf{D}(\mathsf{M})$ consisting of the complexes whose cohomology is concentrated in degree 0.

Proposition 2.2. The obvious functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})^0$ is an equivalence.

This allows us to view M as an additive subcategory of $\mathbf{D}(\mathsf{M})$.

It turns out that the abelian structure of M can be recovered from this embedding.

Proposition 2.3. Consider a sequence

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

in M.

This sequence is exact iff there is a morphism $\gamma : N \to L[1]$ in $\mathbf{D}(\mathsf{M})$ such that

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$$

is a distinguished triangle.

3. Derived Functors

As before M is an abelian category. Recall the localization functor

$$Q : \mathbf{K}(\mathsf{M}) \to \mathbf{D}(\mathsf{M}).$$

It is a triangulated functor, which is the indentity on objects, and inverts quasi-isomorphisms.

Suppose E is some triangulated category, and $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$ a triangulated functor.

We now introduce the right and left derived functors of F. These are triangulated functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathsf{E}$$

satisfying suitable universal properties.

Definition 3.1. A right derived functor of F is a triangulated functor

$$\mathrm{R}F : \mathbf{D}(\mathsf{M}) \to \mathsf{E},$$

together with a morphism

$$\eta : F \to \mathrm{R}F \circ Q$$

of triangulated functors $\mathbf{K}(\mathsf{M}) \to \mathsf{E}$,

satisfying this condition:

(∗) The pair $(\mathrm{R}F, \eta)$ is initial among all such pairs.

Being initial means that if (G, η') is another such pair, then there is a unique morphism of triangulated functors $\theta : \mathrm{R}F \to G$ s.t. $\eta' = \theta \circ \eta$.

The universal condition implies that if a right derived functor $(\mathrm{R}F, \eta)$ exists, then it is unique, up to a unique isomorphism of triangulated functors.

Definition 3.2. A left derived functor of F is a triangulated functor

$$\mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathsf{E},$$

together with a morphism

$$\eta : \mathrm{L}F \circ Q \to F$$

of triangulated functors $\mathbf{K}(\mathsf{M}) \to \mathsf{E}$, satisfying this condition:

(∗) The pair $(\mathrm{L}F, \eta)$ is terminal among all such pairs.

Again, if $(\mathrm{L}F, \eta)$ exists, then it is unique up to a unique isomorphism.

There are various modifications. One of them is a contravariant triangulated functor $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$.

This can be handled using the fact that $\mathbf{K}(\mathsf{M})^{\mathrm{op}}$ is triangulated, and $F : \mathbf{K}(\mathsf{M})^{\mathrm{op}} \to \mathsf{E}$ is covariant.

We will also want to derive bifunctors. Namely to a bitriangulated bifunctor

$$F : \mathbf{K}(\mathsf{M}) \times \mathbf{K}(\mathsf{M}') \to \mathsf{E}$$

we will want to associate bitriangulated bifunctors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \times \mathbf{D}(\mathsf{M}') \to \mathsf{E}\,.$$

This is done similarly, and I won't give details.

4. Resolutions

Consider an additive functor $F : \mathsf{M} \to \mathsf{M}'$ between abelian categories, and the corresponding triangulated functor $\mathbf{K}(F) : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{M}')$, as in Example 1.2.

By slight abuse we write F instead of $\mathbf{K}(F)$. We want to construct (or prove existence) of the derived functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathbf{D}(\mathsf{M}').$$

If F is exact, then $\mathrm{R}F = \mathrm{L}F = F$. (This is an easy exercise.)

Otherwise we need resolutions.

The DG structure of $\mathbf{C}(\mathsf{M})$ gives, for every $M, N \in \mathbf{C}(\mathsf{M})$, a complex of abelian groups $\mathrm{Hom}_{\mathsf{M}}(M, N)$.

Recall that a complex N is called acyclic if $\mathrm{H}^i(N) = 0$ for all i.

Definition 4.1.

1. A complex $I \in \mathbf{K}(\mathsf{M})$ is called K-injective if for every acyclic $N \in \mathbf{K}(\mathsf{M})$, the complex $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is also acyclic.
2. Let $M \in \mathbf{K}(\mathsf{M})$. A K-injective resolution of M is a quasi-isomorphism $M \to I$ in $\mathbf{K}(\mathsf{M})$, where I is K-injective.
3. We say that $\mathbf{K}(\mathsf{M})$ has enough K-injectives if every $M \in \mathbf{K}(\mathsf{M})$ has some K-injective resolution.

Theorem 4.2. If $\mathbf{K}(\mathsf{M})$ has enough K-injectives, then every triangulated functor $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$ has a right derived functor $(\mathrm{R}F, \eta)$.

Moreover, for every K-injective complex $I \in \mathbf{K}(\mathsf{M})$, the morphism $\eta_I : F(I) \to \mathrm{R}F(I)$ in E is an isomorphism.

The proof / construction goes like this: for every $M \in \mathbf{K}(\mathsf{M})$ we choose a K-injective resolution $\zeta_M : M \to I_M$, and we define

$$\mathrm{R}F(M) := F(I_M)$$

and

$$\eta_M := F(\zeta_M) : F(M) \to F(I_M)$$

in E.

Regarding existence of K-injective resolutions:

Proposition 4.3. A bounded below complex of injective objects of M is a K-injective complex.

This is the type of injective resolution used in [Ha].

The most general statement I know is this (see [KS2, Theorem 14.3.1]):

Theorem 4.4. If M is a Grothendieck abelian category, then $\mathbf{K}(\mathsf{M})$ has enough K-injectives.

This includes $\mathsf{M} = \mathsf{Mod}\, A$ for a ring A, and $\mathsf{M} = \mathsf{Mod}\, \mathcal{A}$ for a sheaf of rings $\mathcal{A}$.

Actually in these cases the construction of K-injective resolutions is not so difficult.

Example 4.5. Let $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ be a map of ringed spaces.

(For instance a map of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.)

The map f induces an additive functor

$$f_* : \mathsf{Mod}\,\mathcal{A}_X \to \mathsf{Mod}\,\mathcal{A}_Y$$

called push-forward, which is usually not exact (it is left exact though).

Since $\mathbf{K}(\mathsf{Mod}\,\mathcal{A}_X)$ has enough K-injectives, the right derived functor

$$\mathrm{R}f_* : \mathbf{D}(\mathsf{Mod}\,\mathcal{A}_X) \to \mathbf{D}(\mathsf{Mod}\,\mathcal{A}_Y)$$

exists.

For $\mathcal{M} \in \mathsf{Mod}\,\mathcal{A}_X$ we can use a injective resolution $\mathcal{M} \to \mathcal{I}$ (in the "classical" sense), and therefore

$$\mathrm{H}^q(\mathrm{R}f_*(\mathcal{M})) = \mathrm{H}^q(f_*(\mathcal{I})) = \mathrm{R}^q f_*(\mathcal{M}),$$

where the latter is the "classical" right derived functor.

Analogously we have:

Definition 4.6.

1. A complex $P \in \mathbf{K}(\mathsf{M})$ is called K-projective if for every acyclic $N \in \mathbf{K}(\mathsf{M})$, the complex $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is also acyclic.
2. Let $M \in \mathbf{K}(\mathsf{M})$. A K-projective resolution of M is a quasi-isomorphism $P \to M$ in $\mathbf{K}(\mathsf{M})$, where P is K-projective.
3. We say that $\mathbf{K}(\mathsf{M})$ has enough K-projectives if every $M \in \mathbf{K}(\mathsf{M})$ has some K-projective resolution.

Theorem 4.7. If $\mathbf{K}(\mathsf{M})$ has enough K-projectives, then every triangulated functor $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$ has a left derived functor $(\mathrm{L}F, \eta)$.

Moreover, for every K-projective complex $P \in \mathbf{K}(\mathsf{M})$, the morphism $\eta_P : \mathrm{L}F(P) \to F(P)$ in E is an isomorphism.

The construction of $\mathrm{L}F$ is by K-projective resolutions.

Proposition 4.8. A bounded above complex of projective objects of M is a K-projective complex.

Proposition 4.9. Let A be a ring. Then $\mathbf{K}(\mathsf{Mod}\,A)$ has enough K-projectives.

The concepts of K-injective and K-projective complexes were introduced by Spaltenstein [Sp] in 1988. At about the same time other authors (Keller [Ke], Bockstedt-Neeman [BN], ...) discovered these concepts independently, with other names (such as homotopically injective complex).

Example 4.10. Suppose $\mathbb{K}$ is a commutative ring and A is a $\mathbb{K}$-algebra (i.e. A is a ring and there is a homomorphism $\mathbb{K} \to \mathrm{Z}(A)$).

Consider the bi-additive bifunctor

$$\mathrm{Hom}_A(-,-) : (\mathsf{Mod}\,A)^{\mathrm{op}} \times \mathsf{Mod}\,A \to \mathsf{Mod}\,\mathbb{K}.$$

We have seen how to extend this functor to complexes (this is sometimes called "product totalization"), giving rise to a bitriangulated bifunctor

$$\mathrm{Hom}_A(-,-) : \mathbf{K}(\mathsf{Mod}\,A)^{\mathrm{op}} \times \mathbf{K}(\mathsf{Mod}\,A) \to \mathbf{K}(\mathsf{Mod}\,\mathbb{K}).$$

The right derived functor

$$\mathrm{R}\,\mathrm{Hom}_A(-,-) : \mathbf{D}(\mathsf{Mod}\,A)^{\mathrm{op}} \times \mathbf{D}(\mathsf{Mod}\,A) \to \mathbf{D}(\mathsf{Mod}\,\mathbb{K})$$

can be constructed / calculated by a K-injective resolution in either the first or the second argument.

(cont.) Namely given $M, N \in \mathsf{K}(\mathsf{Mod}\, A)$ we can choose a K-injective resolution $N \to I$, and let

$$\mathrm{R\,Hom}_A(M, N) := \mathrm{Hom}_A(M, I) \in \mathsf{D}(\mathsf{Mod}\, \mathbb{K}). \tag{4.11}$$

Or we can choose a K-injective resolution $M \to P$ in $\mathsf{K}(\mathsf{Mod}\, A)^{\mathrm{op}}$, which is really a K-projective resolution $P \to M$ in $\mathsf{K}(\mathsf{Mod}\, A)$, and let

$$\mathrm{R\,Hom}_A(M, N) := \mathrm{Hom}_A(P, N) \in \mathsf{D}(\mathsf{Mod}\, \mathbb{K}). \tag{4.12}$$

The two complexes (4.11) and (4.12) are canonically related by the quasi-isomorphisms

$$\mathrm{Hom}_A(P, N) \to \mathrm{Hom}_A(P, I) \leftarrow \mathrm{Hom}_A(M, I).$$

If $M, N \in \mathsf{Mod}\, A$ then of course

$$\mathrm{H}^q\big(\mathrm{R\,Hom}_A(M, N)\big) = \mathrm{Ext}^q_A(M, N),$$

where the latter is "classical" Ext.

K-projective and K-injective complexes are good also for understanding the structure of $\mathsf{D}(\mathsf{M})$.

Proposition 4.13. Suppose $P \in \mathsf{K}(\mathsf{M})$ is K-projective and $I \in \mathsf{K}(\mathsf{M})$ is K-injective.

Then for any $M \in \mathsf{K}(\mathsf{M})$ the homomorphisms

$$Q : \mathrm{Hom}_{\mathsf{K}(\mathsf{M})}(P, M) \to \mathrm{Hom}_{\mathsf{D}(\mathsf{M})}(P, M)$$

and

$$Q : \mathrm{Hom}_{\mathsf{K}(\mathsf{M})}(M, I) \to \mathrm{Hom}_{\mathsf{D}(\mathsf{M})}(M, I)$$

are bijective.

Let us denote by $\mathsf{K}(\mathsf{M})_{\text{K-prj}}$ and $\mathsf{K}(\mathsf{M})_{\text{K-inj}}$ the full subcategories of $\mathsf{K}(\mathsf{M})$ on the K-projective and the K-injective complexes respectively.

Corollary 4.14. The triangulated functors

$$Q : \mathsf{K}(\mathsf{M})_{\text{K-prj}} \to \mathsf{D}(\mathsf{M})$$

and

$$Q : \mathsf{K}(\mathsf{M})_{\text{K-inj}} \to \mathsf{D}(\mathsf{M})$$

are fully faithful.

Exercise 4.15. Let $\mathbb{K}$ be a nonzero commutative ring and $A := \mathbb{K}[t]$ the polynomial ring. We view $\mathbb{K}$ as an A-module via $t \mapsto 0$. Find a nonzero morphism $\chi : \mathbb{K} \to \mathbb{K}[1]$ in $\mathsf{D}(\mathsf{Mod}\, A)$. Show that $\mathrm{H}^q(\chi) = 0$ for all $q \in \mathbb{Z}$.

5. DG Algebras

(This is a new section)

A DG algebra (or DG ring) is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, with differential d of degree 1, satisfying the graded Leibniz rule

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i a \cdot \mathrm{d}(b)$$

for $a \in A^i$ and $b \in A^j$.

A left DG A-module is a left graded A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with differential d of degree 1, satisfying the graded Leibniz rule.

Denote by $\mathsf{DGMod}\, A$ the category of left DG A-modules.

As in the ring case, for any $M, N \in \mathsf{DGMod}\, A$ there is a complex of $\mathbb{Z}$-modules $\mathrm{Hom}_A(M, N)$, and

$$\mathrm{Hom}_{\mathsf{DGMod}\, A}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_A(M, N)\big).$$

The homotopy category is $\mathbf{K}(\mathsf{DGMod}\, A)$, with

$$\mathrm{Hom}_{\mathbf{K}(\mathsf{DGMod}\, A)}(M, N) = \mathrm{H}^0\big(\mathrm{Hom}_A(M, N)\big).$$

After inverting the quasi-isomorphisms in $\mathbf{K}(\mathsf{DGMod}\, A)$ we obtain the derived category $\mathbf{D}(\mathsf{DGMod}\, A)$. These are triangulated categories.

Example 5.1. Suppose A is a ring (i.e. $A^i = 0$ for $i \neq 0$). Then $\mathsf{DGMod}\, A = \mathbf{C}(\mathsf{Mod}\, A)$ and $\mathbf{D}(\mathsf{DGMod}\, A) = \mathbf{D}(\mathsf{Mod}\, A)$.

Derived functors are defined as in the ring case, and there are enough K-injectives, K-projective and K-flats in $\mathbf{K}(\mathsf{DGMod}\, A)$.

Let $A \to B$ be a homomorphism of DG algebra. There are additive functors

$$B \otimes_A - : \ \mathsf{DGMod}\, A \rightleftarrows \mathsf{DGMod}\, B \ : \mathrm{rest}_{B/A},$$

where $\mathrm{rest}_{B/A}$ is the forgetful functor. These are adjoint.

We get induced derived functors

$$B \otimes^{\mathrm{L}}_A - : \ \mathbf{D}(\mathsf{DGMod}\, A) \rightleftarrows \mathbf{D}(\mathsf{DGMod}\, B) \ : \mathrm{rest}_{B/A}, \qquad (5.1)$$

where $\mathrm{rest}_{B/A}$ is the forgetful functor. These are adjoint.

Proposition 5.2. If $A \to B$ is a quasi-isomorphism, then the functors (5.1) are equivalences.

Let $f : A \to B$ be a DG algebra homomorphism. A K-flat DG algebra resolution of B relative to A is a factorization of f into $A \xrightarrow{g} \tilde{A} \xrightarrow{h} B$, where h is a quasi-isomorphism, and $\tilde{A}$ is a K-flat DG A-module (on both sides).

Example 5.3. Take $A = \mathbb{Z}$ and $B := \mathbb{Z}/(6)$. We can take $\tilde{A}$ to be the Koszul complex

$$\tilde{A} := (\cdots 0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to 0 \cdots)$$

concentrated in degrees -1 and 0.

Example 5.4. The derived Hochschild cohomology of B relative to A is the cohomology of the complex

$$\mathrm{R}\,\mathrm{Hom}_{\tilde{B} \otimes_A \tilde{B}}(B, B),$$

where $\tilde{B}$ is a resolution as above.

6. Commutative Dualizing Complexes

I will talk about dualizing complexes over commutative rings.

There is a richer theory for schemes, but there is not enough time for it. See [Ha], [Ye2], [Ne1], [Ye4], [AJL], [LH] and their references.

Let A be a noetherian commutative ring. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ the subcategory of $\mathbf{D}(\mathsf{Mod}\, A)$ consisting of bounded complexes whose cohomologies are finitely generated A-modules. This is a full triangulated subcategory.

A complex $M \in \mathbf{D}(\mathsf{Mod}\, A)$ is said to have finite injective dimension if it has a bounded injective resolution. Namely there is a quasi-isomorphism $M \to I$ for some bounded complex of injective A-modules I.

Note that such I is a K-injective complex.

Take any $M \in \mathbf{D}(\mathsf{Mod}\, A)$. Because A is commutative we have a triangulated functor

$$\mathrm{R}\,\mathrm{Hom}_A(-, M) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A).$$

Cf. Example 4.10.

Definition 6.1. A dualizing complex over A is a complex $R \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ with finite injective dimension, such that the canonical morphism

$$A \to \mathrm{R}\,\mathrm{Hom}_A(R, R)$$

in $\mathbf{D}(\mathsf{Mod}\, A)$ is an isomorphism.

If we choose a bounded injective resolution $R \to I$, then there is an isomorphism of triangulated functors

$$\mathrm{R}\,\mathrm{Hom}_A(-, R) \cong \mathrm{Hom}_A(-, I).$$

Example 6.2. Assume A is a Gorenstein ring, namely the free module $R := A$ has finite injective dimension.

There are plenty of Gorenstein rings; for instance any regular ring is Gorenstein.

Then $R \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, and the reflexivity condition holds:

$$\mathrm{R}\,\mathrm{Hom}_A(R, R) \cong \mathrm{Hom}_A(R, R) \cong A.$$

We see that the module $R = A$ is a dualizing complex over the ring A.

Here are several important results from [Ha].

Theorem 6.3. (Duality) Suppose R is a dualizing complex over A. Then the triangulated functor

$$\mathrm{R}\,\mathrm{Hom}_A(-, R) : \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$$

is an equivalence.

Theorem 6.4. (Uniqueness) Suppose R and R' are dualizing complexes over A, and $\mathrm{Spec}\, A$ is connected. Then there is an invertible module P and an integer n such that $R' \cong R \otimes_A P[n]$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Theorem 6.5. (Existence) If A has a dualizing complex, and B is a finite type A-algebra, then B has a dualizing complex.

7. Noncommutative Dualizing Complexes

In this section A is a noncommutative noetherian ring. (This is short for: A is not-necessarily-commutative, and left-and-right noetherian.)

For technical reasons we assume that A is an algebra over a field $\mathbb{K}$.

We denote by A^{op} the opposite algebra (the same addition, but multiplication is reversed), and by $A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ the enveloping algebra.

Thus $\mathsf{Mod}\, A^{\mathrm{op}}$ is the category of right A-modules, and $\mathsf{Mod}\, A^{\mathrm{e}}$ is the category of $\mathbb{K}$-central A-bimodules.

Any $M \in \mathsf{Mod}\, A^{\mathrm{e}}$ gives rise to $\mathbb{K}$-linear functors

$$\mathrm{Hom}_A(-, M) : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, A^{\mathrm{op}}$$

and

$$\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) : (\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathsf{Mod}\, A.$$

These functors can be derived, yielding $\mathbb{K}$-linear triangulated functors

$$\mathrm{R}\,\mathrm{Hom}_A(-, M) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{op}})$$

and

$$\mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) : \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A).$$

One way to construct these derived functors is to choose a quasi-isomorphism $M \to I$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{e}})$, with I a complex that is K-injective on both sides, i.e. over A and over A^{op}.

Then

$$\mathrm{R}\,\mathrm{Hom}_A(-, M) \cong \mathrm{Hom}_A(-, I)$$

and

$$\mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(-, I).$$

The reason that we need $\mathbb{K}$ to be a field is to insure that such "bi-K-injective" resolutions $M \to I$ exist.

Note that even if A is commutative, this setup is still meaningful – not all A-bimodules are A-central!

Definition 7.1. ([Ye1]) A noncommutative dualizing complex over A is a complex $R \in \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ satisfying these three conditions:

(i) The cohomology modules $\mathrm{H}^q(R)$ are finitely generated over A and over A^{op}.

(ii) The complex R has finite injective dimension over A and over A^{op}.

(iii) The canonical morphisms

$$A \to \mathrm{R}\,\mathrm{Hom}_A(R, R)$$

and

$$A \to \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(R, R)$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ are isomorphisms.

Condition (ii) implies that R has a "bounded bi-injective resolution", namely there is a quasi-isomorphism $R \to I$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{e}})$, with I a bounded complex of bimodules that are injective on both sides.

Theorem 7.2. (**Duality**, [Ye1]) Suppose R is a noncommutative dualizing complex over A. Then the triangulated functor

$$\mathrm{R}\,\mathrm{Hom}_A(-, R) : \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$$

is an equivalence, with quasi-inverse $\mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, R)$.

Existence and uniqueness are much more complicated than in the noncommutative case. I will talk about them later.

Example 7.3. The noncommutative ring A is called Gorenstein if the bimodule A has finite injective dimension on both sides.

It is not hard to see that A is Gorenstein iff it has a noncommutative dualizing complex of the form $P[n]$, for some integer n and invertible bimodule P.

Here invertible bimodule is in the sense of Morita theory, namely there is another bimodule $P^{\vee}$ such that

$$P \otimes_A P^{\vee} \cong P^{\vee} \otimes_A P \cong A$$

in $\mathsf{Mod}\, A^{\mathrm{e}}$.

Any regular ring is Gorenstein.

For more results about noncommutative Gorenstein rings see [Jo] and [JZ].

8. Tilting Complexes and Derived Morita Theory

Let A and B be noncommutative $\mathbb{K}$-algebras.

Suppose $M \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ and $N \in \mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$.

The left derived tensor product

$$M \otimes^{\mathrm{L}}_{B} N \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}})$$

exists.

It can be constructed by choosing a resolution $P \to M$ in $\mathbf{K}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, where P is a complex that's K-projective over B^{op};

or by choosing a resolution $Q \to N$ in $\mathbf{K}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$, where Q is a complex that's K-projective over B.

Here is a definition generalizing the notion of invertible bimodule. It is due to Rickard [Ri1], [Ri2].

Definition 8.1. A complex

$$T \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

is called a two-sided tilting complex over A-B

if there exists a complex

$$T^{\vee} \in \mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$$

such that

$$T \otimes^{\mathrm{L}}_{B} T^{\vee} \cong A$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, and

$$T^{\vee} \otimes^{\mathrm{L}}_{A} T \cong B$$

in $\mathbf{D}(\mathsf{Mod}\, B^{\mathrm{e}})$.

When $B = A$ we say that T is a two-sided tilting complex over A.

The complex $T^{\vee}$ is called the inverse of T. It is unique up to isomorphism in $\mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. Indeed we have this result:

Proposition 8.2. Let T be a two-sided tilting complex.

1. The inverse $T^{\vee}$ is isomorphic to $\mathrm{R}\,\mathrm{Hom}_{A}(T, A)$.
2. T has a bounded bi-projective resolution $P \to T$.

Definition 8.3. The algebras A and B are said to be derived Morita equivalent if there is a $\mathbb{K}$-linear triangulated equivalence

$$\mathbf{D}(\mathsf{Mod}\, A) \approx \mathbf{D}(\mathsf{Mod}\, B).$$

Theorem 8.4. ([Ri2]) The $\mathbb{K}$-algebras A and B are derived Morita equivalent iff there exists a two-sided tilting complex over A-B.

Here is a result relating dualizing complexes and tilting complexes.

Theorem 8.5. (**Uniqueness**, [Ye3]) Suppose R and R' are noncommutative dualizing complexes over A.

Then the complex

$$T := \mathrm{R}\,\mathrm{Hom}_{A}(R, R')$$

is a two-sided tilting complex over A, and

$$R' \cong R \otimes^{\mathrm{L}}_{A} T$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

It is easy to see that if T_1 and T_2 are two-sided tilting complexes over A, then so is $T_1 \otimes^{\mathrm{L}}_A T_2$.

This leads to the next definition.

Definition 8.6. ([Ye3]) Let A be a noncommutative $\mathbb{K}$-algebra.

The derived Picard group of A is the group $\operatorname{DPic}(A)$ whose elements are the isomorphism classes (in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$) of two-sided tilting complexes.

The multiplication is

$$[T_1] \cdot [T_2] := [T_1 \otimes^{\mathrm{L}}_A T_2],$$

and the inverse is

$$[T]^{-1} := \mathrm{R}\operatorname{Hom}_A(T, A).$$

Here is a consequence of Theorem 8.5.

Corollary 8.7. Suppose the noncommutative $\mathbb{K}$-algebra A has at least one dualizing complex.

Then the right action

$$[R] \cdot [T] := [R \otimes^{\mathrm{L}}_A T]$$

of the group $\operatorname{DPic}(A)$ on the set of isomorphism classes of dualizing complexes is simply transitive.

It is natural to ask about the structure of the group $\operatorname{DPic}(A)$.

Theorem 8.8. ([RZ], [Ye3]) If the ring A is either commutative (with connected spectrum) or local, then

$$\operatorname{DPic}(A) \cong \operatorname{Pic}(A) \times \mathbb{Z}.$$

Here $\operatorname{Pic}(A)$ is the noncommutative Picard group of A, made up of invertible bimodules.

For nonlocal noncommutative rings the group $\operatorname{DPic}(A)$ is bigger. See the paper [MY] for some calculations. These calculations are related to CY-dimensions of some rings; cf. Example 9.7.

9. Rigid Dualizing Complexes

The material in this final section is largely due to Van den Bergh [VdB1]. His results were extended by J. Zhang and myself.

Again A is a noetherian noncommutative algebra over a field $\mathbb{K}$, and $A^{\mathrm{e}} = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$.

Take $M \in \mathsf{Mod}\, A^{\mathrm{e}}$. Then the $\mathbb{K}$-module $M \otimes_{\mathbb{K}} M$ has four commuting actions by A, which we arrange as follows.

The algebra $A^{\mathrm{e;\,in}} := A^{\mathrm{e}}$ acts on $M \otimes_{\mathbb{K}} M$ by

$$(a_1 \otimes a_2) \cdot_{\mathrm{in}} (m_1 \otimes m_2) := (m_1 \cdot a_2) \otimes (a_1 \cdot m_2),$$

and the algebra $A^{\mathrm{e;\,out}} := A^{\mathrm{e}}$ acts by

$$(a_1 \otimes a_2) \cdot_{\mathrm{out}} (m_1 \otimes m_2) := (a_1 \cdot m_1) \otimes (m_2 \cdot a_2).$$

The bimodule A is viewed as an object of $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ in the obvious way.

Now take $M \in \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. We define the square of M to be the complex

$$\mathrm{Sq}(M) := \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e};\,\mathrm{out}}}(A, M \otimes_{\mathbb{K}} M) \in \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e};\,\mathrm{in}}).$$

We get a functor

$$\mathrm{Sq} : \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}}) \to \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}}).$$

This is not an additive functor. Indeed, it is a quadratic functor: given an element $a \in \mathrm{Z}(A)$ and a morphism $\phi : M \to N$ in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, one has

$$\mathrm{Sq}(a\phi) = \mathrm{Sq}(\phi a) = a^2\, \mathrm{Sq}(\phi).$$

Note that the cohomologies of $\mathrm{Sq}(M)$ are

$$\mathrm{H}^q(\mathrm{Sq}(M)) = \mathrm{Ext}^q_{A^{\mathrm{e}}}(A, M \otimes_{\mathbb{K}} M),$$

so they are precisely the Hochschild cohomologies of $M \otimes_{\mathbb{K}} M$.

A rigid complex over A (relative to $\mathbb{K}$) is a pair (M, ρ) consisting of a complex $M \in \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, and an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}(M)$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Definition 9.1. ([VdB1]) A rigid dualizing complex over A (relative to $\mathbb{K}$) is a rigid complex (R, ρ) such that R is a dualizing complex.

Let (M, ρ) and (N, σ) be rigid complexes over A.

A rigid morphism

$$\phi : (M, \rho) \to (N, \sigma)$$

is a morphism $\phi : M \to N$ in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \rho\ } & \mathrm{Sq}(M) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}(\phi)} \\ N & \xrightarrow{\ \sigma\ } & \mathrm{Sq}(N) \end{array}$$

is commutative.

Theorem 9.2. (**Uniqueness**, [VdB1], [Ye3]) Suppose (R, ρ) and (R', ρ') are both rigid dualizing complexes over A. Then there is a unique rigid isomorphism

$$\phi : (R, \rho) \xrightarrow{\simeq} (R', \rho').$$

As for existence, let me first give an easy case.

Proposition 9.3. If A is finite over its center, and is finitely generated as $\mathbb{K}$-algebra, then A has a rigid dualizing complex.

Actually, in this case it is quite easy to write down a formula for the rigid dualizing complex.

In the next existence result, by a filtration $F = \{F_i(A)\}_{i \in \mathbb{Z}}$ of the algebra A we mean an ascending exhaustive nonnegative filtration.

Such a filtration gives rise to a graded $\mathbb{K}$-algebra

$$\mathrm{gr}^F(A) = \bigoplus_{i \geq 0} \mathrm{gr}^F_i(A).$$

Theorem 9.4. (**Existence**, [VdB1], [YZ3]) Suppose A admits a filtration F such that $\mathrm{gr}^F(A)$ is finite over its center and finitely generated as $\mathbb{K}$-algebra. Then A has a rigid dualizing complex.

This theorem applies to the ring of differential operators $\mathcal{D}(C)$, where C is a smooth commutative $\mathbb{K}$-algebra (and $\mathrm{char}\, \mathbb{K} = 0$).

It also applies to any quotient of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$.

I will finish with some examples.

Example 9.5. Let A be a noetherian $\mathbb{K}$-algebra satisfying these two conditions:

- A is smooth, namely the A^{e}-module A has finite projective dimension.
- There is an integer n such that

$$\mathrm{Ext}^q_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} A & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

Then A is a regular ring, and the complex $R := A[n]$ is a rigid dualizing complex over A.

Such an algebra A is called an n-dimensional Artin-Schelter regular algebra, or an n-dimensional Calabi-Yau algebra.

Example 9.6. Let $\mathfrak{g}$ be an n-dimensional Lie algeba, and $A := \mathrm{U}(\mathfrak{g})$, the universal enveloping algebra.

Then the rigid dualizing complex of A is $R := A^{\sigma}[n]$, where A^{σ} is the trivial bimodule A, twisted on the right by an automorphism σ.

Using the Hopf structure of A we can express A^{σ} like this:

$$A^{\sigma} \cong \mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \textstyle\bigwedge^n \mathfrak{g},$$

the twist by the 1-dimensional representation $\bigwedge^n \mathfrak{g}$.

So A is a twisted Calabi-Yau algebra.

If $\mathfrak{g}$ is semi-simple then there is no twist, so A is Calabi-Yau. This was used by Ven den Bergh in his duality for Hochschild (co)homology [VdB2].

Example 9.7. Let

$$A := \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$$

the 2×2 matrix algebra.

The rigid dualizing complex here is

$$R := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}).$$

It is known that

$$R \otimes^{\mathrm{L}}_A R \otimes^{\mathrm{L}}_A R \cong A[1]$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

So A is a Calabi-Yau algebra of dimension $\frac{1}{3}$.

- END -

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