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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

**Speaker's Name:** Graham Leuschke

**Talk Title:** Non-commutative desingularizations and MCM modules I

**Date:** 01 / 31 / 13 **Time:** 11 : 00 am / pm (circle one)

**List 6-12 key words for the talk:** maximal Cohen-Macaulay modules, noncomm desingularizations, crepant resolution, Nakamura's G-Hilbert scheme.

**Please summarize the lecture in 5 or fewer sentances:** Describe recent work on noncomm. resolutions of singularities with a focus on the role of maximal Cohen-Macaulay modules (MCM). In particular, Auslander's algebraic McKay correspondence between MCM modules, skew group rings of the binary polyhedral groups were discussed, leading to Van den Bergh's definition of a noncomm. crepant resolution.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- <u>**Computer Presentations**</u>: Obtain a copy of their presentation
- <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
- <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

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01/31/13
11:00 am

Graham Leuschke:
"Non-commutative desingularizations
and MCM modules I"

Goal: Illustrate the idea [and the (potential) definitions] of noncommutative desingularizations with concrete examples, concentrating on the role of MCM modules.

- MCM McKay Correspondence:

Setup: $S = k[[x_1, \dots, x_n]]$, char. $k = 0$, $n \geq 2$

$\underset{\text{finite group}}{G} \subseteq GL_n(k)$ acting on $S$

$R = S^G$, invariant subring

Observations:

+ $R$ is a complete Cohen Macaulay (CM) normal domain
+ $S$ is maximal CM (MCM) as an $R$-module [$\operatorname{depth}_R S = n$ or $S$ is f.g. free over a power series subring of $R$].

My goal: Understand $S$ as $R$-module, since it's a "God-given" MCM module.

To study $S$, we introduce the "twisted group ring" $S \# G$ (semi-direct product, or smash product), it is:

+ free $S$-module on $\{\sigma\}_{\sigma \in G}$
+ twisted multiplication defined by:

$$(s\sigma)(t\tau) = s\sigma(t)\sigma\tau \quad , \quad \forall s,t \in S,\ \sigma,\tau \in G$$

Observation: An $S \# G$-module is just an $S$-module with a compatible $G$-action.

Easy computation : $\mathrm{Hom}_{S\#G}(-,-) \cong \mathrm{Hom}_S(-,-)^G$

$\xRightarrow{\operatorname{char} k = 0}$ $\mathrm{Ext}^i_{S\#G}(-,-) \cong \mathrm{Ext}^i_S(-,-)^G$

Consequencely,

$$\mathrm{gl.dim}\ S\#G = n \ (= \dim S)$$

- There is a natural map $\gamma : S\#G \longrightarrow \mathrm{End}_R(S)$ , which is a ring homomorphism $s\sigma \longmapsto s\sigma$, as endomorphism $s\sigma : t \mapsto s.\sigma(t)$

<u>Theorem</u>: [Auslander '62] : If $G$ is <u>small</u> (that is, contains no elements fixing a hyperplane point-wise), then $\gamma$ is an isomorphism.
"pseudo - reflection"

- From now on, assume $G$ is small.

<u>Corollary</u>: $R$ has a module - finite algebra (ie. finitely generated as a, module) namely $S\#G \cong \mathrm{End}_R(S)$, which has finite global dimension and .... (other things)

<u>Q</u>: What should [....] mean if we want to call $\mathrm{End}_R(S)$ a non comm. desingularization ?

There's more, we have:

$$\begin{tikzcd}
S\#G \arrow[r, "\gamma", "\cong"'] \arrow[d, "\cong", "s\sigma \mapsto \sigma^{-1}(s)\sigma^{-1}"'] & \mathrm{End}_R(S) \\
(S\#G)^{op} \arrow[r, "\cong", "s\sigma \mapsto \text{right mult. by } s\sigma"'] & \mathrm{End}_{S\#G}(S\#G) \arrow[u, "\text{rest}"']
\end{tikzcd}$$

where "rest" means restriction to $S \cong S.\frac{1}{|G|}\sum \sigma \subseteq S\#G$

<u>Corollary</u>: These are all isomorphisms, so [looking at idempotents on the right hand side (RHS)]:

$$\operatorname{proj} S\#G = \operatorname{add}_R(S)$$
$$P \longleftrightarrow P^G$$

S#G direct summands of S#G

direct summands of direct sums of S as R-mod. (Recall: Our goal!)

Even more: The representation theory of G is also involved, via

$$\begin{array}{ccccc} \operatorname{rep}_k G & \longrightarrow & \operatorname{proj} S\#G & \longrightarrow & \operatorname{add}_R(S) \\ W & \longmapsto & S \otimes_k W & \longmapsto & (S \otimes_k W)^G \\ & & \text{G acting diag.} & & \text{covariants} \end{array}$$

The composition is <u>not</u> an equivalence of categories but is a bijection on objects.

– <u>Specialize to $n=2$</u>:

+ <u>Herzog '78</u>: $\operatorname{MCM}(R) = \operatorname{add}_R(S)$ (always $\supseteq$)

So in particular, R has <u>finite CM type</u>, that is, only finitely many indecomposable MCM modules $/ \cong$ (up to isom.)

And so, if $n=2$, there are finitely many indecomposable MCM (= reflexive) modules $M_1, \ldots, M_\ell$, and the "Auslander algebra":

$$\Lambda = \operatorname{End}_R(M_1 \oplus \ldots \oplus M_\ell)$$ has finite gl.dim and is MCM/R

+ Finite CM Type (FCMT) fails if $n \geq 3$, (except $k[[x,y,z]]^{\mathbb{Z}_2}$).

+ There are other examples of FCMT in dim $\geq 3$, and for those the Auslander algebra still has finite global dim, <u>but</u> is not as nice as in dim $= 2$, it's not MCM as an R-module.

– <u>A little geometry</u>: (Artin - Verdier)

Assume $n=2$, and set $X = \operatorname{Spec} R$, $G \leq GL_2(k)$

Then $X$ has a unique "minimal" resolution of singularities $\pi : Y \to X$ with exceptional fiber a bunch of $\mathbb{P}$'s.

There is a 1-1 correspondence between indecomposable MCM $R$-mods and the components of the exceptional fiber.

- There is a <u>derived</u> version of things we've covered:
  The minimal resolution of singularities is realized as <u>Nakamura's $G$-Hilbert scheme</u>:

$$\begin{aligned} \mathcal{H} &:= \mathrm{Hilb}_G(\mathbb{C}^2) \\ &= \{ I \mid \mathbb{C}[[x,y]]/I \text{ is isom. to } \mathbb{C}G \text{ as } G\text{-rep'ns.} \} \subseteq S \end{aligned}$$

<u>Theorem</u>: [Kapranov - Vasserot '00] Let $G \subseteq SL_2$, $\exists$ equiv.

$$\begin{tikzcd} D^b(\mathrm{coh}\, \mathcal{H}) \arrow[r, "\simeq"] & D^b(\mathrm{coh}_G\, \mathbb{C}^2) \arrow[d, "\simeq \text{ always}"] \\ & D^b(\mathrm{mod}\ S\#G) \end{tikzcd}$$

$D^b(\mathrm{coh}_G\, \mathbb{C}^2)$ = $G$-equivariant coherent sheaves

So $S\#G$ doesn't just <u>know</u> about the resolution of singularities, it <u>is</u> the resolution.

<u>Q</u>: Does Kapranov - Vasserot extend to dim $\geq 3$?
In dim $\geq 3$, $\mathcal{H} = \mathrm{Hilb}_G(\mathbb{C}^n)$ makes sense, <u>but</u>
+ If $n \geq 3$, it's not a <u>minimal</u> resolution of singularities but is <u>crepant</u>.
+ If $n \geq 4$, it's not even smooth.

<u>Theorem</u>: [Bridgeland - King - Reid '01]: $G \subseteq SL_3(k)$, then $\exists$ equiv.

$$D^b(\mathrm{coh}\, \mathcal{H}) \xrightarrow{\simeq} D^b(\mathrm{coh}_G\, \mathbb{C}^3) \xrightarrow[\text{always}]{\simeq} D^b(\mathrm{mod}\ S\#G) = D^b(\mathrm{mod}\ \mathrm{End}_R(S))$$

So even though $S$ no longer contains all MCM's, $\Lambda = \mathrm{End}_R(S)$ still "is" the resolution.

<u>Conjecture</u>: [BKR, Douglas, Nakamura]: $G \leq SL_n(\mathbb{C})$

If $\pi: Y \longrightarrow \mathbb{C}^n/G$ is a crepant resolution of singularities, then $D^b(\mathrm{coh}\, Y) \xrightarrow{\simeq} D^b(\mathrm{coh}_G(\mathbb{C}^n)) \cong D^b(\mathrm{mod}\, S \# G)$

In particular, this implies

$D^b(\mathrm{coh}\, Y)$ is independent of $Y$.

Take all of these as motivation for:

<u>Defn</u>: [Van den Bergh]

Let $R$ be an Gorenstein normal domain. A <u>non-commutative crepant resolution</u> of $R$ (NCCR) is an $R$-algebra $\Lambda$ which:

+ is of finite global dim.
+ $\Lambda \cong \mathrm{End}_R(M)$, where ${}_RM$ is reflexive [MCM?]
+ $\Lambda$ is MCM as an $R$-module

<u>Remark</u>: There is no general relationship between $M$ being MCM and $\mathrm{End}_R(M)$ being MCM.

<u>One useful criterion</u>: $d = \dim R$, $M$ is MCM over CM local and $\mathrm{Ext}_R^{1,2,\dots,d-2}(M,M) = 0 \implies \mathrm{End}_R(M)$ is MCM.

($\Leftarrow$ if $R$ is an isolated sing.)

- <u>First theorems and conjectures</u>:

<u>Thm</u> (Van den Bergh) Let $R$ be Gorenstein normal, $X = \mathrm{Spec}\, R$, $\pi: Y \to X$ a resolution of sings. Assume $\pi$ is crepant, the fibers of $\pi$ have dim $\leq 1$ and exceptional locus has dim $\geq 2$. Then:

$R$ has a NCCR, $\Lambda = \mathrm{End}_R(M)$ with $M$ is MCM, and $D^b(\mathrm{coh}\, Y) \cong D^b(\mathrm{mod}\, \Lambda)$.

If in addition, $\dim R = 3$, then all the crepant resolutions (NCCR's and usual ones) are derived equivalent.