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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Jason Bell

Talk Title: Introduction to Growth and growth function II

Date: 02/01/13 **Time:** 9:15 am / pm (circle one)

List 6-12 key words for the talk: GK dim, nilpotent-by-finite, exponential growth, (lower) transcendence degree, periodic group

Please summarize the lecture in 5 or fewer sentances: State and build up tools to prove Gromov's Theorem for f.g. group with finite GKdim K[G]. Propose some questions about f.g. periodic group. Define isoperimetric profile and properties of (lower) transcendence degree that lead to the classification of division algebras of Ld = 2, relating to Artin's conjecture.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- Computer Presentations: Obtain a copy of their presentation
- Overhead: Obtain a copy or use the originals and scan them
- Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- Handouts: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

02/01/13 Jason Bell :
9:15am "Introduction to Growth and growth functions II"

Gromov's Theorem:

let G be a f.g. group and suppose $k[G]$ has finite GK dim, then $\exists\, N \trianglelefteq G$, $[G:N] < \infty$, N nilpotent.

Converse was known before, ie. if G f.g. nilpotent-by-finite $\Rightarrow$ $k[G]$ has finite GK dim.

- [Bass - Guivarc'h]
If G is f.g. nilpotent, $G = G_1 \trianglerighteq G_2 = G' \trianglerighteq G_3 = [G_1, G_2] \trianglerighteq \ldots \trianglerighteq G_{k+1} = 1$
$\Rightarrow$ GKdim $k[G] = \sum_{i=1}^{k} i\, rk(G_i/G_{i+1})$

e.g. $G = \langle x,y,z \mid [x,y]=z, [z,x]=1, [z,y]=1 \rangle$
GK dim $\mathbb{C}[G] = 4$

- Solvable case:
Facts: 1) If G is f.g. solvable group, then either G has exponential growth or it is nilpotent-by-finite
2) If $k[G]$ is also noetherian, then G is polycyclic-by-finite (ie. all subgroups are f.g.)

Tits alternative:

$\underset{\text{f.g}}{G} \leq GL_n(k)$ k is a field

$\swarrow$ G is solvable-by-finite (*)

$\searrow$ free subgroup on 2 generators $\Rightarrow$ exponential growth

(*) $\Rightarrow$ nilpotent-by-finite or exp. growth

Black-box:

let G f.g., $S = S^{-1}$, $1 \in S$, $\# S < \infty$, S generates G

Create $\mathbb{C}[G]$-module V from S, G such that:

1) $\dim V = 1$, G acts trivially on V if $\# G < \infty$
2) $\dim V < \infty$ if G has poly. bounded growth
3) If G is infinite and $N \leq G$ acts trivially on $V \Rightarrow [G:N] = \infty$

"Proof of Gromov's Thm":

$\varphi : G \longrightarrow GL(V)$, image of φ is nilpotent-by-finite

GKdim $k[G] \geq$ GKdim $k[N] + 1$, for $N \leq G$ s.t. $\begin{array}{c} G \\ |\infty \\ N \end{array}$

We may assume N to be nilpotent-by-finite

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

$\divideontimes$. $\square$

Corollary 1:

If G is f.g. and G has poly. bounded growth,
$\Rightarrow k[G]$ is Noetherian.

Corollary 2: If G is a f.g. periodic group of poly. bounded growth,
$\Rightarrow G$ is finite.

Grigorchuk: constructed periodic groups of intermediate growth $e^{o(\sqrt{n})}$

Questions:

1) Can one show that a periodic f.g. group whose growth function is $e^{o(\sqrt{n})}$ is finite?
2) If $k[G]$ is Noetherian and G is periodic
$\Rightarrow G$ is finite?

Possibility for 1) : PBW theorem

If $\mathcal{L}$ is a Lie algebra $\Rightarrow$ $\mathcal{U}(\mathcal{L})$ has either finite integer GK dim or growth $\geq \exp(c\sqrt{n})$ for some $c>0$

$$x_n \in S^n \setminus S^{n-1}, \quad \{ x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \mid \varepsilon_i \in \{0,1\} \}$$

- Burnside

  G f.g. periodic

  $\Downarrow$?

  G is finite

- Kurosh

  A f.g. k-algebra, k field, algebraic/k

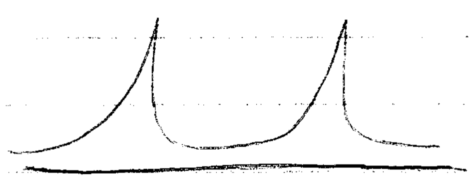
  $\Downarrow$?

  $\dim_k A < \infty$

Answer: NO

Lenagan - Smoktunowicz found counter-examples of finite GKdim to Kurosh.

- Bottleneck method: $k\{x,y\}/([x,y])$

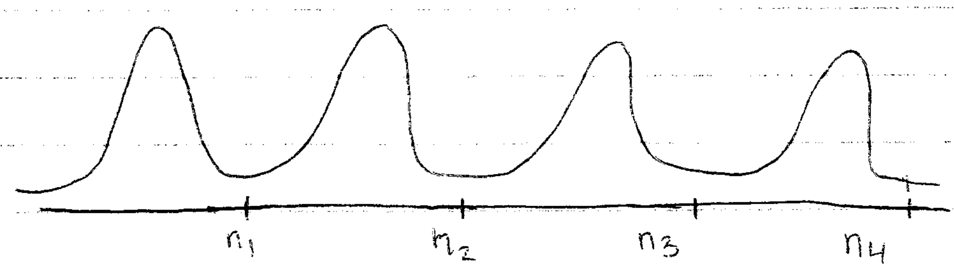


When field k is countable

$A = k\{x,y\} = k \oplus A_1 \oplus A_2 \oplus \cdots$

$f_1, f_2, f_3, \ldots$ enumeration of $\bigoplus_{i\geq 1} A_i$

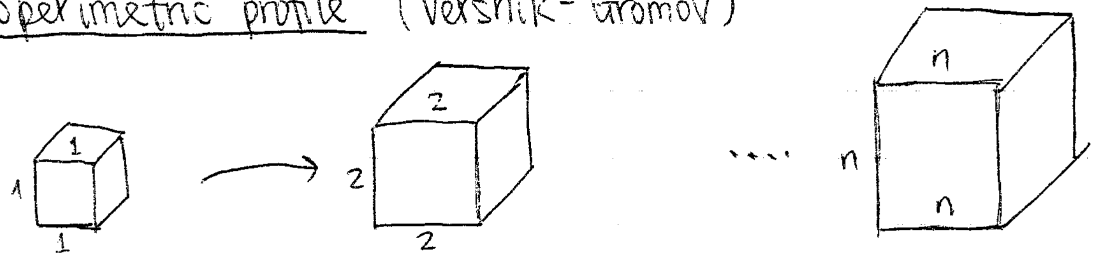
$n_1 < n_2 < n_3 < \cdots$ sparse



Impose relations of degree n_1 (homogeneous) such that f_1^N becomes zero mod these relations.

$\rightsquigarrow$ Continue similarly w/ $n_2, n_3, \dots$

- <u>Isoperimetric profile</u> (Vershik-Gromov)



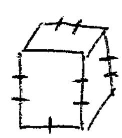
V_1 $\qquad$ $V_2 = 8 \cdot V_1 = 2^3 V_1$ $\qquad$ $V_n \sim n^3$

$S_n \sim 6n^2 \sim V_n^{2/3}$

$S \sim V^{\frac{d-1}{d}}$

- If G is a f.g. group
  $\# S < \infty$, $1 \in S$, $S = S^{-1}$
  $X \subseteq G$
  $\partial_S(X) := \#(SX \setminus X)$ boundary of X wrt. S

  <u>ex</u>: $G = \mathbb{Z}^3$
  $S = \{(0,0,0), (1,0,0), \dots\}$
  $X = ([-n,n] \cap \mathbb{Z})^3$



Isoperimetric profile $\rightarrow$ $I_0(n; G, S) := \inf\limits_{|X|=n} \partial_S(X) = n^{\frac{d-1}{d}}$ if GKdim $G = d$

$I_0(n+m; G, S) \leq I_0(n; G, S) + I_0(m; G, S)$

- <u>Lower Transcendence degrees</u>:
  let D be a division ring over a field k
  $1 \in W \subseteq D$ finite dimensional over k
  $V \subseteq D \rightsquigarrow \partial_W(V) = \dim(WV/V)$

If $\exists\, W$ and $c > 0$ s.t. $\partial_W(V) \geq c(\dim V)^{\frac{d-1}{d}}$

$\Rightarrow$ D has lower transcendence degree $\geq d$

(Ld = lower trans. degree)

$$Ld(D) = \sup_d \{\exists\, W,\ c>0 \text{ s.t. } \partial_W(V) \geq c(\dim V)^{\frac{d-1}{d}},\ \forall\, V \subseteq D\}$$

+ If A is a domain of GKdim $= d$ and $D = Q(A)$ quotient

$\Rightarrow Ld(D) \leq d$.

- <u>What should transcendence degree T satisfy?</u>

1) $T_K(K) = \operatorname{trdeg}_K K$
2) $T(Q(A)) \leq \operatorname{GKdim} A$
3) $\begin{array}{c} D \\ | \\ E \end{array}$, $T(E) \leq T(D)$ and if D/E is ∞ dim'l, D f.g. $\Rightarrow T(D) \geq T(E) + 1$

Ld satisfies 1), 2), and partially 3): $Ld(E) \leq Ld(D)$
but not sure if $Ld(D) \geq Ld(E) + 1$?

+ Artin's conjecture $\subseteq$ Classification of division algebras of $Ld(D) = 2$

finite dim'l over center

$K(x;\sigma)$

$K(x;\delta)$

Sklyanian division rings

<u>Q</u>: Is there a "f.g." division ring D of $Ld_{\mathbb{C}}(D) = 2$ that is not on Artin's list?

- <u>Another topic</u>: $\mathbb{C}[x,y][t;\sigma^*]$, where $\sigma \in \operatorname{Aut}_{\mathbb{C}}(\mathbb{A}^2)$

$x \mapsto \alpha x + \beta y$ $\qquad$ $x \mapsto x + p(y)$

$y \mapsto \gamma x + \delta y$ $\qquad$ $y \mapsto y$

If we take an affine variety X and embed

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{A}^N \\ \sigma \downarrow & & \downarrow \tau \\ X & \xrightarrow{i} & \mathbb{A}^N \end{array}$$

N large

τ tame
