MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen Email/Phone: van.nguyen3@gmail.com

Speaker's Name: Maria Chlouveraki

Talk Title: Symplectic reflection algebras II

Date: 02/01/13 Time: 3:45 am / pm (circle one)

List 6-12 key words for the talk: rational Cherednik algebras, Euler element, Hecke algebras, canonical basic set

Please summarize the lecture in 5 or fewer sentances: Focus on a particular class of symplectic reflection algebras: the rational Cherednik algebras Study their representation theory and its connection with the representation theory of Hecke algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

02/01/13 Maria Chlouveraki:

3:45pm "Symplectic reflection algebras II"

Rational Cherednik algebras:

$W \subset GL(\mathfrak{h})$ complex reflection group

$V = \mathfrak{h} \oplus \mathfrak{h}^*$

$(\ ,\ ) : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ such that $(y,x) = x(y)$

ω standard symplectic form on V, that is,

$$\omega((y_1,x_1),(y_2,x_2)) = (y_1,x_2) - (y_2,x_1)$$

S = set of symplectic reflections of (V,ω,W)

= set of complex reflections of $(W,\mathfrak{h})$

$\mathcal{A}$ = set of reflecting hyperplanes of W

For $s \in S$, let

H_s = the reflecting hyperplane of s

$\alpha_s \in \mathfrak{h}^*$ such that $\operatorname{Ker}\alpha_s = H_s$, (Basis of $\operatorname{Im}(s-\mathrm{id}_V)|_{\mathfrak{h}^*}$)

$\alpha_s^\vee \in \mathfrak{h}$ such that $(\alpha_s^\vee, \alpha_s) = 1-\det(s)$, (Basis of $\operatorname{Im}(s-\mathrm{id}_V)|_{\mathfrak{h}}$)

$\underline{c} : S \to \mathbb{C}$ conjugacy invariant function

$$H_{t,\underline{c}} := (TV^* \rtimes W) \Big/ \begin{array}{l} [x_1,x_2]=0,\ [y_1,y_2]=0 \\ [y,x] = t(y,x) - 2\displaystyle\sum_{s\in S} \frac{\underline{c}(s)}{1-\det(s)} (y,\alpha_s)(\alpha_s^\vee,x)s \\ \\ x, x_1, x_2 \in \mathfrak{h}^* \text{ and } y, y_1, y_2 \in \mathfrak{h} \end{array}$$

Case $t \neq 0$, $t=1$:

$Z(H_{1,\underline{c}}) = \mathbb{C}$, very few f.d. modules

PBW $\Rightarrow$ $H_{1,\underline{c}} \underset{\text{V.S.}}{\cong} \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]$

$\underline{\text{Category}}$ $\mathcal{O}$ = category of f.g. $H_{1,\underline{c}}$-modules
locally nilpotent for the action of $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$
($M \in \mathcal{O} \leftrightarrow \forall m \in M, \exists N \gg 0$ s.t. $\mathfrak{h}^N . m = 0$)

<u>Rmk</u>: Every $M \in \mathcal{O}$ is f.g. as a $\mathbb{C}[\mathfrak{h}]$-module
(Ginzburg) $\quad \mathrm{supp}_{\mathbb{C}[\mathfrak{h}]}(M) = \underset{\text{"full"}}{\mathfrak{h}}$ or $\subseteq \bigcup\limits_{H \in \mathcal{A}} H$
reflecting hyperplane

- $\mathcal{O}$ is a highest weight category:
  + <u>Standard (Verma) modules</u>:
    $\Delta(E)_{E \in \mathrm{Irr}(W)}$ $\quad \Delta(E) = \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*] \rtimes W}^{H_{1,\underline{c}}}(E)$
    irreducibles
  + <u>Simple modules</u>:
    $L(E) = \underset{\text{simple}}{\mathrm{Head}}(\Delta(E))$

    $\{L(E)\}_{E \in \mathrm{Irr}(W)}$ is a complete set of pairwise non-isom. simple modules.

  + <u>Ordering on standard modules</u>: $\rightsquigarrow \mathrm{Irr}(W)$: $E < F$
    Decomposition matrix $D_{\mathcal{O}} = \left([\Delta(F) : L(E)]\right)_{E, F \in \mathrm{Irr}(W)}$
    a) $[\Delta(E) : L(E)] = 1$
    b) If $[\Delta(F) : L(E)] \neq 0$, then either $E = F$ or $E < F$

Example of an ordering c-function:
- <u>Euler element</u>:

$$eu = \sum_{i=1}^{n} x_i y_i \underbrace{\left(- \sum_{s \in S} \frac{2\underline{c}(s)}{1 - \det(s)} . s \right)}_{= eu_W}$$

basis of $\mathfrak{h}^*$ $\quad$ dual basis of $\mathfrak{h}$

$eu_W \in Z(\mathbb{C}W)$

c_E = scalar with which eu_W acts on representation E

$E < F \Rightarrow c_F - c_E \in \mathbb{Z}_{>0}$

<u>Rmk</u>: We can take $-c_E$

$P(E)$ = projective cover $(L(E))$

<u>BGG reciprocity</u> : $(P(E) : \Delta(F)) = [\Delta(F) : L(E)]$

<u>KZ-functor</u> : Exact functor

$$\mathcal{O} \longrightarrow \mathcal{H}_\theta(W)\text{-mod}$$

↖ Hecke algebra

$\mathcal{H}$ = Hecke algebra of W

= deformation of $\mathbb{C}W$ over $\mathbb{C}\left[(q_{s,j}^{\pm 1}) \;;\; s \in S,\ 1 \leq j < \text{order}(s)\right] =: R$

conj. invariant

eg. $\mathcal{H}(\mathfrak{S}_n) = \left\langle T_1, \dots, T_{n-1} \;\middle|\; \begin{array}{l} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ i = 1, \dots, n-2 \\ T_iT_j = T_jT_i \text{ for } |i-j| > 1 \\ (T_i - q)(T_i + 1) = 0,\ \text{for } i = 1, \dots, n-1 \end{array} \right\rangle$

(under good assumptions)

$$\mathrm{Irr}(\mathbb{C}(q_{s,j})\mathcal{H}) \longleftrightarrow \mathrm{Irr}(\mathbb{C}W)$$

$\theta : R \longrightarrow \mathbb{C}$ ring homomorphism

$q_{s,j} \longmapsto \exp(2\pi i k_{s,j})$

$k_{s,j} \in \mathbb{C}$

Decomposition matrix $D_{\mathcal{H}} = \underbrace{\left(\quad\right)}_{\mathrm{Irr}(\mathcal{H}_\theta(W))} \Big\}\ \mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\mathrm{Frac}(R)\mathcal{H})$

$$\text{KZ-exact} \Rightarrow [\Delta(F) : L(E)] = [KZ(\overset{\neq 0}{\Delta(F)}) : \overset{\text{simple}}{KZ(L(E))}]$$
$$\underline{\text{if}} \quad KZ(L(E)) \neq 0$$

<u>Prop.</u> $\mathcal{O}$ semisimple $\Leftrightarrow \Delta(E) = L(E), \ \forall E$

$\Leftrightarrow D_{\mathcal{O}} = D_{\mathcal{H}}$ identity matrix

$\Leftrightarrow \mathcal{H}_{\mathcal{O}}(W)$ semisimple

$\exists$ specific semisimplicity criterion for $\mathcal{H}_{\mathcal{O}}$

$\mathcal{H}_{\mathcal{O}}$ is semisimple unless $\mathcal{O}(q_{s,j})$ is a root of unity

$q_{s,j} \longleftrightarrow k_{s,j} \in \mathbb{Q}$

<u>Corollary</u>: For generic $\underline{c}$, $\mathcal{O}$ is semisimple.

$$\mathbb{B} = \{E \in \mathrm{Irr}(W) \mid KZ(L(E)) \neq 0\}$$

(*) <u>Prop.</u>

a) $\{KZ(L(E)) \mid E \in \mathbb{B}\}$ is a complete set of pairwise non-isom. simple $\mathcal{H}$-mod

b) $[KZ(\Delta(E)) : KZ(L(E))] = 1$, for $E \in \mathbb{B}$

c) If $[KZ(\Delta(E)) : KZ(L(F))] \neq 0$ for $F \in \mathrm{Irr}(W)$, $E \in \mathbb{B}$, then either $E = F$ or $E < F$

$$D_{\mathcal{O}} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ * & & \ddots & \\ & & & 1 \end{pmatrix} \rightsquigarrow D_{\mathcal{H}_{\mathcal{O}}} = \begin{pmatrix} 1 & & 0 \\ * & \ddots & 1 \\ \hline * * & \cdots & * \end{pmatrix} \Big\} \mathbb{B} \quad \Big\downarrow \text{increasing ordering}$$

<u>Q</u>: $\mathbb{B} = ?$

$$\begin{tikzcd} \mathcal{O} \arrow[r, "KZ"] \arrow[d] & \mathcal{H}_{\mathcal{O}}\text{-mod} \\ \mathcal{O}/\mathcal{O}_{tor} \arrow[ur] & \end{tikzcd}$$

$\mathcal{O}_{tor}$ ← torsion

, where $\mathcal{O}_{tor}$ = modules that do not have full support

$$\mathrm{supp}(M) \subseteq \bigcup_{H \in \mathcal{A}} H$$

K.L. theory implies that prop (*) holds when W is a Weyl group for the ordering induced by Lusztig's functions

Then $\mathbb{B}$ is called a <u>canonical basic set</u>.

<u>Q</u>: Does a canonical basic set for all complex reflection group?

↳ Yes. For $G(\ell, p, n)$ some exceptional case. (†)

- If a function Good for $\mathcal{O}$
  - + the existence of canonical basic sets (in a uniform way)
  - + the description for $\mathbb{B}$ in the cases where canonical basic sets have been explicitly constructed (†).

<u>Theorem</u>: (C-Gordon-Griffeth)

If W is a finite coxeter group or of finite $G(\ell, 1, n)$, then $KZ(L(E)) \neq 0 \iff E$ belongs to canonical basic set $\mathbb{B}$

eg: S_n $\qquad q \mapsto \exp\left(\frac{2\pi i}{e}\right), \quad e > 1$

$\mathrm{Irr}(S_n) \longleftrightarrow \{\lambda \mid \lambda \vdash n\}$

$KZ(L(\lambda)) \neq 0 \iff \lambda$ is an e-regular partition

$\nexists\, i$ such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e-1} \neq 0$

<u>Case $t=0$</u>:

All simple modules are finite dim'l.

$$\mathcal{U}_{0,\underline{c}} = eH_{0,\underline{c}}e = Z(H_{0,\underline{c}})$$

$$\mathcal{U}_{0,\underline{c}} \supseteq \left(\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W\right) =: \mathfrak{m}$$

Restricted Rational Cherednik algebras :

$$\overline{H}_{0,\underline{c}} = H_{0,\underline{c}} / \mathfrak{m}_+ H_{0,\underline{c}}$$

where $\mathfrak{m}_+$ = ideal of $\mathfrak{m}$ of elements with zero constant term

PBW $\Rightarrow$ $\overline{H}_{0,\underline{c}} \underset{V.S.}{\cong} \mathbb{C}[\mathfrak{h}]^{coinv.} \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]^{coinv.}$ ← coinvariant algebra

$\mathbb{C}[\mathfrak{h}]^{coinv.} = \mathbb{C}[\mathfrak{h}] / \langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle$

$\dim \overline{H}_{0,\underline{c}} = |W|^3$

+ Baby-Verma modules : $(\overline{\Delta}(E))_{E \in Irr(W)}$

$$\overline{\Delta}(E) := \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*]^{coinv.} \rtimes W}^{\overline{H}_{0,\underline{c}}} E$$

+ Simple modules : $\overline{L}(E)$ = Head of $\overline{\Delta}(E)$

$\{\overline{L}(E)\}_{E \in Irr(W)}$ is a complete set of pairwise nonisom. simple modules

+ Calogero - Moser partition :
Partition of $\overline{H}_{0,\underline{c}}$ into blocks

$$\overline{H}_{0,\underline{c}} = \bigoplus_{i=1}^{K} B_i$$

$1 = b_1 + \dots + b_K$, $\forall L$ simple , $\exists! \, i_L$ such that $b_{i_L} L \neq 0$
L, L' belong to the same block $\Leftrightarrow i_L = i_{L'}$
$\rightarrow$ partition of $Irr(W)$

- W = Weyl group , $\mathcal{H}(W)$ Hecke algebra
Partition of $Irr(W)$ into Lusztig's families $\equiv$ Rouquier families

For generic $\underline{c}$, C-M partition $\equiv$ Rouquier families

$G(\ell, 1, n)$: Martino

$G(\ell, p, n)$, G_4 : Bellamy

$G_5, G_6, G_8, G_{10}, G_{23}, G_{24}, G_{26}$: Thiel

Not true for G_{25}

Bonnafé – Rouquier : definition of K-L cells for complex ref. gps.

$$\mathcal{U}_{0,\underline{c}} = Z(H_{0,\underline{c}}) \overset{\text{PBW}}{\cong} \mathbb{C}[V]^W$$

$$X_{\underline{c}}(W) = \operatorname{Spec} \mathcal{U}_{0,\underline{c}}$$

<u>Theorem</u>: V/W admits a symplectic resolution $\Longleftrightarrow$

$X_{\underline{c}}(W)$ is smooth for generic $\underline{c}$ $\overset{\text{Bellamy–Martino}}{\Longleftrightarrow}$

C-M partition is trivial for generic $\underline{c}$ $\overset{\text{Bellamy}}{\Longleftrightarrow}$

$W = G(\ell, 1, n)$ or $W = G_4$

//