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Speaker's Name: Eleonora Cinti

Talk Title: Pattern formation, optimal transport and interpolation inequalities

Date: 08 / 23 / 2013 Time: 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Optimal transport, interpolation inequality Coarsening rates Branching in superconductors, Kantorovich duality

Please summarize the lecture in 5 or fewer sentances: This lecture provides proofs of weak sense as well as strong sense of interpolation inequalities.

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Pattern formation, optimal transport and interpolation inequalities

Eleonora Cinti

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(joint work with Felix Otto)

MSRI Berkeley, August, 2013

Outline

- Some motivations:

  - Coarsening rates in critical mixture
  - Branching in micromagnetics
  - Branching in superconductors

- Interpolation inequalities in weak form.
- Interpolation inequalities in strong form:

  - Ledoux method
  - Geometric construction.

Coarsening rates

- Configurations: Scalar order parameter

$$u(t,x) : (0,+\infty) \times [0,\Lambda]^d \to \in [-1,1]$$

describes local composition of inhomogeneous binary mixture.

- Free energy: Ginzburg-Landau energy density:

$$E(u) = \fint \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(1-u^2)^2 \, dx.$$

- Regime: Critical mixture, i. e.,

$$\fint u \, dx = 0.$$

Dynamic given by the Cahn-Hilliard equation

$$\partial_t u - \Delta \frac{\partial E}{\partial u} \; = \; 0,$$

- preserves volume fraction $\fint u \, dx = 0$,
- decreases energy $E(u)$.

Coarsening — Generic solutions in simulations

Initial data: $u = 0$ + small amplitude white noise.

Qualitative observations:

- Initial stage: Phases separate and form two domains $\{u \approx 1\}$ and $\{u \approx -1\}$.
- Late stages: Typical length scale ℓ of domains increases with time.

Quantitative observations: $\ell \sim t^{1/3}$

Consider the sharp-interface limit: $u \in \{-1, +1\}$. Approximate energy density:

$$\begin{aligned} E &\approx \text{energy of 1-}d\text{ interfacial layer} \\ &\qquad \times \text{area of sharp interface per system volume} \\ &\sim \fint |\nabla u|\, dx \sim \frac{1}{\text{length}}. \end{aligned}$$

Kohn & Otto: The Cahn-Hilliard equation

$$\partial_t u - \Delta \frac{\partial E}{\partial u} = 0$$

is the gradient flow for E with respect to the Euclidean structure given by $|| \nabla |^{-1} \cdot ||_{L^2}$. Define the length L as the induced distance,

$$L^2 = ⨍ || \nabla |^{-1} u|^2 \, dx,$$

where

$$\int || \nabla |^{-1} u|^2 := \min \left\{ \int |J|^2 : \nabla \cdot J = u \right\} = \int |\nabla \varphi|^2,$$

with φ satisying

$$-\Delta \varphi = u.$$

Kohn & Otto (2002): in a time-average sense $\ell \sim 1/E \gtrsim t^{1/3}$.

Main ingredients in the proof:

- An interpolation estimate:

$$EL \gtrsim C.$$

- An energy dissipation rate:

$$\frac{dE}{dt} \lesssim -\left(\frac{dL}{dt}\right)^2.$$

- An ODE argument.

Branching in micromagnetics

(Hubert - Kohn and Mueller - Choksi, Kohn and Otto - Conti) Magnetization

$$m : \Omega \to S^2$$

Energy:

$$E(m,h) = d^2 \int_\Omega |\nabla m|^2 dx + Q \int_\Omega (m_1^2 + m_2^2) dx + \int_{\mathbb{R}^3} |h|^2 dx$$

where

$$\begin{cases} \nabla \cdot (h+m) = 0 \quad \text{distributionally in } \mathbb{R}^3 \\ \nabla \times h = 0 \quad \text{distributionally in } \mathbb{R}^3 \end{cases}$$

Thus

$$\int_{\mathbb{R}^3} |h|^2 dx = \int_{\mathbb{R}^3} ||\nabla|^{-1} \nabla \cdot m|^2 dx.$$

Consider

$$\Omega = (-l, l)^2 \times (0, t).$$

Heuristic for the branched configuration:

$$E(m, h) \sim Q^{1/3} d^{2/3} l^2 t^{1/3}$$

It is better than the unbranced Ansatz for $t >> Q^{1/2} d$.

Rigorous lower bound

Reduced model:

$$E(m,h) = d^2 \int_\Omega |\nabla' m_3| dx + \int_{\mathbb{R}^3} |h'|^2 dx$$

where

$$\begin{cases} \nabla' \cdot h' + \partial_3 m_3 = 0 \quad \text{distributionally in } \mathbb{R}^3 \\ \nabla \times h' = 0 \quad \text{distributionally in } \mathbb{R}^3 \end{cases}$$

($\Gamma-$convergence-type result by Otto and Viehmann)

Choksi, Kohn and Otto:

$$E(m,h) \geq CQ^{1/3} d^{2/3} l^2 t^{1/3}.$$

Main ingredient: interpolation inequality.

Branching in superconductors

The variational problem has two unknown:

the magnetic flux B

the domain pattern which is described by a function

$\chi : (-l, l)^d \times (-1, 1) \to \{0, 1\}$.

Meissner effect: $(1 - \chi)B = 0$ (no flux in the superconducting phase).

The model is described by the continuity equation

$$\begin{cases} \partial_z \chi + \nabla \cdot (\chi B) = 0 \ \text{ in } \ (-l, l)^d \times (-1, 1) \\ \chi \to \phi \ \text{ as } \ z \to \pm 1, \end{cases} \tag{1}$$

where $\phi > 0$ is a constant that corresponds to the prescribed magnetic flux density.

The energy associated to the system is given by

$$E(\chi) = \int_{-1}^{1} \int_{(-l,l)^d} (|\nabla \chi| + \chi |B|^2) dxdz. \tag{2}$$

Using that

$$W^2(\chi, \phi) \lesssim \int_{-1}^{1} \int_{(-l,l)^d} \chi |B|^2 dxdz,$$

to give a lower bound for the energy it is enough to bound from below the quantity

$$\int_{-1}^{1} \int_{(-l,l)^d} |\nabla \chi| dxdz + W^2(\chi, \phi).$$

.

Interpolation inequalities (in weak form)

Proposition

Let $u : [0, \Lambda]^d \to \mathbb{R}$ satisfy $\int u = 0$, then

$$||u||_{w-L^{4/3}} \leq C||\nabla u||_{L^1}^{1/2}|||\nabla|^{-1}u||_{L^2}^{1/2},$$

where $||u||_{w-L^{4/3}} := \sup_{\mu>0} \mu|\{|u| \geq \mu\}|^{3/4}$ denotes the weak $L^{4/3}-$norm of u.

Interpolation inequalities (in weak form)

Sketch of the proof

Consider

$$\chi = \begin{cases} 1 & \text{for} \quad u \geq \mu \\ 0 & \text{for} \quad u \in (-\mu, \mu) \\ -1 & \text{for} \quad u \leq -\mu \end{cases}$$

and define a convolution kernel

$$K_R(x) = \frac{1}{R^d} K\left(\frac{x}{R}\right),$$

where K is a smooth compactly supported nonnegative function s.t $\int K = 1$.

We have

$$\mu|\{|u|>\mu\}| \leq \int \chi u = \int (u - K_R * u)\chi + \int (K_R * u)\chi.$$

Estimate:

$$\int |u - K_R * u| \leq R \int |\nabla u|,$$

and

$$\begin{aligned} \int (K_R * u)\chi &\leq \left(\int ||\nabla|^{-1} u|^2\right)^{1/2} \left(\int |\nabla (K_R * \chi)|^2\right)^{1/2} \\ &\leq \left(\frac{1}{R^2} \int ||\nabla|^{-1} u|^2 \int \chi\right)^{1/2}. \end{aligned}$$

Use Young inequality and optimize in R.

Interpolation inequalities (in weak form)

Proposition [Conti, Niethammer, Otto]

Let $u : [0, \Lambda]^2 \to \mathbb{R}$ satisfy $u \geq 0$ with $\frac{1}{\Lambda^2} \int u = \Phi$. Suppose $\mu \geq 2\Phi$, then

$$\mu \log^{1/4} \frac{\mu}{\Phi} |\{|u| \geq \mu\}|^{3/4} \leq C ||\nabla u||_{L^1}^{1/2} |||\nabla|^{-1}(u - \Phi)||_{L^2}^{1/2}.$$

(crucial ingredient in the proof of the lower bound of the energy depending on the volume fraction Φ)

Interpolation inequalities (in weak form)

Sketch of the proof

Careful choice of the convolution kernel:

$$K_{R,L}(x) = \begin{cases} \frac{1}{\pi R^2} & \text{if } |x| \leq R \\ \frac{1}{\pi R^2} \frac{\log \frac{L}{|x|}}{\log \frac{L}{R}} & \text{if } R < |x| \leq L \\ 0 & \text{if } |x| > L. \end{cases}$$

Write

$$\begin{aligned} \mu|\{|u| > \mu\}| &\leq \int \chi u \\ &= \int (u - \Phi) \min\{K_{R,L} * \chi, 1\} + \int \Phi \min\{K_{R,L} * \chi, 1\} \\ &\quad + \int u(\chi - \min\{K_{R,L} * \chi, 1\}). \end{aligned}$$

Interpolation inequalities (in weak form)

Proceed as before and use that

$$\int |\nabla \min\{K_{R,L} * \chi, 1\}|^2 \le \frac{2}{R^2}\frac{1}{\log(L/R)}\int \chi.$$

Choose L/R such that

$$\Phi\left(\frac{L}{R}\right)^2 = \frac{1}{2}\mu,$$

use Young inequality and optimize in R.

Interpolation inequalities (in strong form)

Proposition

There exists a constant $C < \infty$ such that for all periodic functions $u : (0, \Lambda)^d \to \mathbb{R}$, $\int u = 0$ we have

$$\|u\|_{L_{4/3}} \leq C \left(\int |\nabla u| \right)^{1/2} \left(\int ||\nabla|^{-1} u|^2 \right)^{1/4}$$

Interpolation inequalities (in strong form)

Sketch of the proof

Following an idea of Ledoux for the proof a similar interpolation inequality we introduce a factor $M \gg 1$ to be adjusted later. We have:

$$\begin{aligned} \int \chi_\mu u &= \int (\chi_\mu - \chi_{\mu,R}) u + \int \chi_{\mu,R} u \\ &= \int_{|u| \leq M\mu} (\chi_\mu - \chi_{\mu,R}) u + \int_{|u|>M\mu} (\chi_\mu - \chi_{\mu,R}) u + \int \chi_{\mu,R} u. \end{aligned}$$

Using that $\|\chi_\mu - \chi_{\mu,R}\|_\infty \leq 2$, we obtain the inequality

$$\begin{aligned} \int_{|u|>\mu} |u| &\leq M\mu \int |\chi_\mu - \chi_{\mu,R}| + 2 \int_{|u|>M\mu} |u| + \int \chi_{\mu,R} u \\ &\leq M\mu R \int |\nabla \chi_\mu| + 2 \int_{|u|>M\mu} |u| + \int \chi_{\mu,R} u. \end{aligned}$$

Interpolation inequalities (in strong form)

We multiply with $\mu^{-\frac{2}{3}}$ and choose $R = \mu^{-\frac{1}{3}}$. Integrating over $\mu \in (0, \infty)$, we get

$$
\begin{aligned}
&\int_0^\infty \mu^{-\frac{2}{3}} \int_{|u|>\mu} |u| dx d\mu \\
&\quad \leq \quad M \int_0^\infty \int |\nabla \chi_\mu| dx d\mu + 2 \int_0^\infty \mu^{-\frac{2}{3}} \int_{|u|>M\mu} |u| dx d\mu \\
&\qquad\quad + \int \int_0^\infty \mu^{-\frac{2}{3}} \chi_{\mu,R} d\mu \, u dx \\
&\quad \leq \quad M \int_0^\infty \int |\nabla \chi_\mu| dx d\mu + 2 \int_0^\infty \mu^{-\frac{2}{3}} \int_{|u|>M\mu} |u| dx d\mu \\
&\qquad\quad + \|\nabla (\int_0^\infty \mu^{-\frac{2}{3}} \chi_{\mu,R} d\mu)\|_2 \, \||\nabla|^{-1} u\|_2,
\end{aligned}
$$

Interpolation inequalities (in strong form)

We have

$$\int_0^{\infty} \mu^{-\frac{2}{3}} \int_{|u(x)|>\mu} |u(x)| dx d\mu = \int |u(x)| \int_0^{|u(x)|} \mu^{-\frac{2}{3}} d\mu dx = 3 \int |u|^{\frac{4}{3}}.$$

and

$$\begin{aligned} &\int_0^{\infty} \mu^{-\frac{2}{3}} \int_{|u(x)|>M\mu} |u(x)| dx d\mu \\ = & \int |u(x)| \int_0^{M^{-1}|u(x)|} \mu^{-\frac{2}{3}} d\mu dx = 3M^{-\frac{1}{3}} \int |u|^{\frac{4}{3}}. \end{aligned}$$

By the coarea formula we get

$$\int_0^{\infty} \int |\nabla \chi_\mu| dx d\mu = \int_0^{\infty} (\mathrm{Per}(\{u > \mu\}) + \mathrm{Per}(\{u < -\mu\})) d\mu = \|\nabla u\|_1.$$

Interpolation inequalities (in strong form)

Integrating by parts and after some computations we get

$$\begin{aligned}||\nabla(\int_0^\infty \mu^{-\frac{2}{3}}\chi_{\mu,R}d\mu)||_2^2 &\leq C\int_0^\infty \mu^{\frac{1}{3}}\ |\{|u|>\mu\}|d\mu \\ &= C\int\int_0^{|u(x)|}\mu^{\frac{1}{3}}d\mu dx \\ &= C\int|u|^{\frac{4}{3}}.\end{aligned}$$

Interpolation inequalities (in strong form)

Collecting all the terms we have

$$3\int |u|^{\frac{4}{3}}$$
$$\leq \quad M\|\nabla u\|_1 + 6M^{-\frac{1}{3}}\int |u|^{\frac{4}{3}} + C\left(\int |u|^{\frac{4}{3}}\right)^{\frac{1}{2}} \||\nabla|^{-1}u\|_2.$$

We obtain the desired estimate by absorbing the middle right-hand side term for $M \gg 1$ and absorbing the first factor of the last right-hand side term by Young's inequality.

Interpolation inequalities (in strong form)

Proposition

There exists a constant $C < \infty$ such that for all periodic functions $u : (0,\Lambda)^2 \to \mathbb{R}$, with $u \geq -1$ and $\frac{1}{\Lambda^2}\int u = 0$, we have

$$\|u \ln^{\frac{1}{4}} \max\{u, e\}\|_{\frac{4}{3}} \;\leq\; C\|\nabla u\|_1^{\frac{1}{2}} \,\||\nabla|^{-1}u\|_2^{\frac{1}{2}}. \tag{3}$$

Geometric Construction

Main ingredient in the proof is the following

GEOMETRIC CONSTRUCTION:

For $\chi(x) \in \{0,1\}$ and $R \ll L$ there exists a potential $\phi_{R,L}(x) \in [0,1]$ such that

$$\int \chi \lesssim R \int |\nabla \chi| + \int \chi \phi_{R,L}, \tag{4}$$

$$\int |\nabla \phi_{R,L}|^2 \lesssim R^{-2}(\ln^{-1} \frac{L}{R}) \int \chi \tag{5}$$

$$\int \phi_{R,L} \lesssim L^2 R^{-2} \int \chi. \tag{6}$$

This type of geometric construction was first used by Choksi, Conti, Kohn, and Otto in the context of branched patterns in superconductors, but its main ingredient goes back to De Giorgi.

Geometric Construction

Steps in the geometric construction:

- Define the set $\Omega_R = \{x \mid |\{\chi = 1\} \cap B_{\frac{R}{2}}(x)| > \frac{1}{2}|B_{\frac{R}{2}}(x)|\}$.
- Show that there exists a finite subset $C \subset \Omega_R$ such that

  $\Omega_R \subset \bigcup_{y \in C} B_R(y)$ while $R^2 \# C \lesssim \int \chi$, where C is maximal with the property that $B_{R/2}(y) \cap B_{R/2}(y') = \emptyset$ for every $y, y' \in C$, $y \neq y'$.

Geometric Construction

We introduce the capacity potential $\hat{\phi}_{R,L}$ of $B_R(0)$ in $B_L(0)$ given by

$$\hat{\phi}_{R,L}(\hat{x}) := \left\{ \begin{array}{ccccccc} 1 & \text{for} & & & |\hat{x}| & \leq & R \\ \frac{\ln \frac{L}{|\hat{x}|}}{\ln \frac{L}{R}} & \text{for} & R & \leq & |\hat{x}| & \leq & L \\ 0 & \text{for} & L & \leq & |\hat{x}| & & \end{array} \right\} \in [0,1].$$

We define

$$\phi_{R,L}(x) := \max_{y \in C} \hat{\phi}_{R,L}(x-y) \ \in \ [0,1].$$

Geometric Construction

With this choice, $\phi_{R,L}$ satisfies

- $$\int \chi \lesssim R \int |\nabla \chi| + \int \chi \phi_{R,L}.$$
- $$\int \phi_{R,L} \lesssim L^2 R^{-2} \int \chi.$$
- $$\int |\nabla \phi_{R,L}|^2 \lesssim R^{-2} (\ln^{-1} \frac{L}{R}) \int \chi$$

Other inequalities...

The geometric contruction is used also in the proof of two other interpolation inequalities (crucial ingredient in the proof of a lower bound for the energy in superconductors):

$$||u||_{L^{\frac{3d+2}{3d}}} \leq C||\nabla u||_{L^1}^{\frac{2d}{3d+2}} W(u,1)^{\frac{d}{3d+2}},$$

where W denotes the Wasserstein distance.

Ingredients in the proof:

- geometric construcion,
- Kantorovich duality for W.

Other inequalities...

The Wasserstein distance is given by

$$W^2(u,v) := \inf\left\{\int\int |x-y|^2 d\pi(x,y) \,\middle|\, \int d\pi(\cdot,y) = u,\ \int d\pi(x,\cdot) = v\right\}.$$

The measure on the product space π is called *transportation plan* and it is admissible if its projections to first and second coordinates are measures with densities u and v respectively.

A useful property of the Wasserstein distance is the following *Kantorovich duality*:

$$W^2(u,v) = \sup\left\{\int u(x)\phi(x)dx + \int v(y)\psi(y)dy \,\middle|\, \phi(x)+\psi(y) \le |x-y|^2\right\}.$$

Other inequalities...

Sketch of the proof

Step 1. We carry out the geometric construction in any dimension d. Given a function $\chi : [0,\Lambda]^d \to \{0,1\}$ there exists a set Ω_R and a potential $\phi_R(x) \in \{0,1\}$ such that

$$\Omega_R \subset \bigcup_{y \in C} B_R(y) \quad \text{and} \quad \#C \lesssim \frac{1}{R^d} \int \chi,$$

where C is maximal with the property that $B_{R/2}(y) \cap B_{R/2}(y') = \emptyset$ for every $y, y' \in C$, $y \neq y'$.

$$\phi_R = 1 \text{ in } \Omega_R, \quad \phi_R = 0 \text{ in } \mathbb{R}^d \setminus \Omega_R.$$

$$\int \chi \lesssim R \int |\nabla \chi| + \int \chi \phi_R.$$

Other inequalities...

Step 2. Using the geometric construction as before we write

$$\begin{aligned}\int \chi_\mu &\lesssim R\int |\nabla \chi_\mu| + \int \chi_\mu \phi_{\mu,R} \\ &\lesssim R\int |\nabla \chi_\mu| + \frac{1}{\mu}\int \phi_{\mu,R} u. \end{aligned} \tag{7}$$

We multiply (**??**) by $\mu^{(2+3d)/(3d)}$, we choose $R = \mu^{-2/(3d)}$ and we integrate in $\int \frac{d\mu}{\mu}$, to get

$$\begin{aligned}\int \mu^{\frac{2+3d}{3d}} \int \chi_\mu dx \frac{d\mu}{\mu} &\lesssim \int_0^{+\infty} \int |\nabla \chi_\mu| dx d\mu \\ &+ \int \left(\int_0^{+\infty} \mu^{\frac{2}{3d}} \phi_{\mu,R}(x) \frac{d\mu}{\mu} \right) u(x) dx. \end{aligned}$$

Using the coarea formula as before, the first term becomes

$$\int_0^{+\infty} \int |\nabla \chi_\mu| dx d\mu = ||\nabla u||_1.$$

Other inequalities...

Step 3. We use the Kantorovich duality to estimate the second term on the r.h.s. . We set $\phi(x) := \int_0^{+\infty} \mu^{\frac{2}{3d}} \phi_{\mu,R}(x) \frac{d\mu}{\mu}$. We have that

$$\int \phi(x) u(x) dx \leq W^2(u,1) + \int \psi(y) dy,$$

where

$$\psi(y) = \sup_x \{\phi(x) - |x-y|^2\} = \sup_x \left\{ \int_0^{+\infty} \mu^{\frac{2}{3d}} \phi_{\mu,R}(x) \frac{d\mu}{\mu} - |x-y|^2 \right\}.$$

Other inequalities...

Another interpolation inequality arising in superconductors:

$$\begin{aligned} &|| \max\{u, \nu^{\frac{3d+1}{3d+3}}\}||_{L^{\frac{3d+3}{3d+1}}} \\ &\leq C||\nabla u||_{L^1}^{\frac{2d}{3d+3}} \left(\inf_{v \geq 0} \left\{ \nu^{\frac{2}{d+1}} W^2(u, v) + \nu^{\frac{1-d}{d+1}} ||v||_{H^{-1/2}} \right\} \right)^{\frac{1}{3}}. \end{aligned}$$

Ingredients in the proof:

- geometric construction,
- Kantorovich duality for W,
- $H^{1/2}$-estimates.