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NOTETAKER CHECKLIST FORM

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Name: Shuangjian Zhang Email/Phone: shuangjian.zhang@mail.utoronto.ca

Speaker's Name: Marina Chugunova

Talk Title: Mass concentration phenomena for the long-wave unstable thin-film equation

Date: 08 / 23 / 2013 Time: 02 : 00 am / pm (circle one)

List 6-12 key words for the talk: thin-film equation, Lubrication equation.

Please summarize the lecture in 5 or fewer sentences: This lecture provides introduction to thin-film equation and lubrication equation, as well as proof of finite speed of propagation for the range $0<n<\frac{1}{2}$, $n/2 < m < 6-n$.

CHECK LIST

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Mass concentration phenomena for the long-wave unstable thin-film equation.

Marina Chugunova

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Collaboration:

Mary Pugh (University of Toronto)
and Roman Taranets (The University of Nottingham)

MSRI Workshop, Berkeley
Connections for Women on Optimal Transport, August 23, 2013

Weak and strong generalized solutions.

[Bernis and Friedman, 1990]

$$u_t = -(f(u)\,u_{xxx})_x,\;\; f(u) \sim |u|^n,\;\; u_x(\pm a, t) = u_{xxx}(\pm a, t) = 0,$$
$$P = \overline{Q}_T(\{u = 0\} \cup \{t = 0\}),\;\; \Omega = (-a, a).$$

- **weak generalized solution**

$$\iint_{Q_T} u\phi_t + \iint_P f(u) u_{xxx}\phi_x = 0,$$
$$u \in C^{1/2,1/8}_{x,t}(\overline{Q}_T), \qquad f(u)u_{xxx} \in L^2(P).$$

- **strong generalized solution**

$$\iint_{Q_T} u\phi_t - \iint_{Q_T} f(u)u_{xx}\phi_{xx} - \iint_{Q_T} f'(u)u_x u_{xx}\phi_x = 0,$$
$$u \in L^2(0, T; H^2(\Omega)).$$

Initial values and boundary conditions.

[Bernis and Friedman, 1990]

$$u_t = -(u^n u_{xxx})_x, \;\; u_x(\pm a) = u_{xxx}(\pm a) = 0, \;\; \Omega = (-a, a)$$

$$P = \overline{Q}_T(\{u = 0\} \cup \{t = 0\})$$

Initial values and boundary conditions:

- $u(x, 0) = u_0(x), \quad x \in \overline{\Omega}$ **and** $u_x(., t) \to u_{0x}$ **strongly in** $L^2(\Omega)$ **as** $t \to 0.$

- $u_x(\pm a) = u_{xxx}(\pm a) = 0$ **at all points of the lateral boundary where** $\{h \neq 0\}$**.**

Some results for the case n = 1.

[Bernis, Peletier, Williams 1991, Otto, 1998; Carrillo, Toscani, 2002; Carlen and Ulusoy, 2007; Mattes, McCann, Savare, 2009;]

$$u_t = -(u\, u_{xxx})_x,\ \ x \in R^1,\ \ t > 0, \qquad u(x,0) = u_0(x) \geq 0$$

Explicit self-similar source type solution:

$$u(x,t) = t^{-1/5} \left(\frac{1}{120} (a^2 - t^{-2/5} x^2)_+^2 \right).$$

The equation defines a gradient flow $u_t = \left[u \left(\frac{\delta E}{\delta u} \right)_x \right]_x$.
$\frac{\delta E}{\delta u}$ **denotes the** L^2**-gradient of** $E(u) = 1/2 \int u_x^2(x) dx$.
Metric is the optimal transportation distance.
For non-negative initial data u_0 **that belongs to** $H^1(R)$ **and also has a finite mass and second moment, the strong solution converges in** H^1 **norm to the unique self-similar source type solution.**

Research motivation.

Lubrication equation:

We study nonnegative weak solutions of long-wave unstable lubrication equation

$$h_t = -(f(h)\, h_{xxx})_x - (g(h)\, h_x)_x$$

with power-law coefficients $f(h) = a_0\, h^n$ **and** $g(h) = a_1\, h^m$ **that become singular in finite time (** $h(x,t)$ **gives the hight of the evolving free-surface). The exponent** n **plays stabilizing role due to fourth-order forward diffusion term and the exponent** m **plays destabilizing role due to backward second-order diffusion term.**

Modeling of crystal growth: **One of approaches to modeling strongly anisotropic crystal and epitaxial growth is using regularized, anisotropic Cahn-Hilliard-type equations (4th order nonlinear PDE:** $h_t + (M(h)(h_{xxx} + h_x))_x = 0$ **where mobility** $M(h) = h(1-h) \sim h$**). Such problems arise during the growth and coarsening of thin films.**

Long-wave instability.

Long-wave unstable lubrication equation

$$h_t = -a_0 \left(|h|^n \, h_{xxx} \right)_x - a_1 \left(|h|^m \, h_x \right)_x,$$

where $a_0 > 0$, $a_1 > 0$, and h is real valued.
Perturbing a constant steady state slightly,

$$h_0(x) = \bar{h} + \epsilon h_1(x, 0) = \bar{h} + \epsilon \cos(\xi x + \phi),$$

and linearizing the equation about $\bar{h}$, the small perturbation $h_1(x, t)$ will (approximately) satisfy

$$h_t = -a_0 |\bar{h}|^n h_{xxxx} - a_1 |\bar{h}|^m h_{xx}.$$

Hence the constant steady state is linearly unstable to long wave perturbations:

$$\xi^2 < |\bar{h}|^{m-n} a_1 / a_0 \quad \to \quad h_1(x, t) \sim e^{-a_0 \xi^2 |\bar{h}|^n \left(\xi^2 - \frac{a_1}{a_0} |\bar{h}|^{m-n} \right) t} \cos(\xi x + \phi)$$

Axillary functionals.

Lubrication equation:

Long-wave unstable lubrication equation

$$h_t = -(f(h)\, h_{xxx})_x - (g(h)\, h_x)_x$$

with power-law coefficients $f(h) = a_0\, h^n$ **and** $g(h) = a_1\, h^m$**.**

Energy functional:

$$E(T) := \int\limits_{\Omega} \tfrac{a_0}{2} h_x^2(x,T) - a_1 D_0(h(x,T))\, dx, \;\; D_0(z) := \frac{z^{m-n+2}}{(m-n+1)(m-n+2)}$$

Entropy functional: $S(T) := \int\limits_{\Omega} G(u(x,T))\, dx$

$$G(z) := \begin{cases} \frac{z^{-n+2}}{(-n+2)(-n+1)} \textbf{if } n \neq \{1,2\}, \\ z \ln z - z \textbf{ if } n = 1, \\ -\ln z \textbf{ if } n = 2. \end{cases} ; \;\; (G(z))'' = \frac{1}{z^n}.$$

Existence, finite speed and blow-up results.

Long-wave unstable lubrication equation

$$h_t = -a_0 \left(|h|^n h_{xxx}\right)_x - a_1 \left(|h|^m h_x\right)_x,$$

where $a_0 > 0$, $a_1 > 0$, and h is real valued.

The main results are:

- **short-time existence of nonnegative strong solutions on Ω given nonnegative initial data**
- **finite speed of propagation for these solutions if their initial data had compact support within Ω**
- **finite-time blow-up for solutions of the Cauchy problem that have initial data with negative energy**

Short-time existence.

Given nonnegative initial data that has finite entropy, we prove the short-time existence of a nonnegative weak solution if $n > 0$ and $m \geqslant n/2$. (a short-time result for $n > 0$ and $m \geqslant n$ was known)

[Sketch of the proof] Given $\delta, \varepsilon > 0$, a regularized parabolic problem is considered:

$$h_t + \left(f_{\delta\varepsilon}(h)(a_0 h_{xxx} + a_1 D''_\varepsilon(h) h_x)\right)_x = 0,$$
$$\frac{\partial^i h}{\partial x^i}(-a,t) = \frac{\partial^i h}{\partial x^i}(a,t) \text{ for } t > 0,\ i = \overline{0,3},$$
$$h(x,0) = h_{0,\varepsilon}(x)$$

where

$$f_{\delta\varepsilon}(z) := f_\varepsilon(z) + \delta = \frac{|z|^{4+n}}{|z|^4 + \varepsilon |z|^n} + \delta,\ \ D''_\varepsilon(z) := \frac{|z|^{m-n}}{1+\varepsilon|z|^{m-n}}, \epsilon > 0, \delta > 0.$$

Approximation of initial data.

For $\epsilon > 0$, the nonnegative initial data, h_0, is approximated via

$$h_0 + \varepsilon^\theta \leq h_{0,\varepsilon} \in C^{4+\gamma}(\overline{\Omega}) \text{ for some } 0 < \theta < \tfrac{2}{5},$$

$$\frac{\partial^i h_{0,\varepsilon}}{\partial x^i}(-a) = \frac{\partial^i h_{0,\varepsilon}}{\partial x^i}(a) \text{ for } i = \overline{0,3},$$

$$h_{0,\varepsilon} \to h_0 \text{ strongly in } H^1(\Omega) \text{ as } \varepsilon \to 0.$$

Finite speed of propagation.

We were successful in proving finite speed of propagation for the range $0 < n \leq 1/2$**,** $n/2 \leq m < 6 - n$ **and for the range** $1/2 < n < 3$**,** $n/2 \leq m < 3\,n + 4$**.**

If supp$(h_0) \subseteq [-r_0, r_0] \subset (-a, a)$ **then there is a nondecreasing function** $\Gamma(t)$ **and a time** T_{speed} **such that supp**$(h(\cdot, t)) \subseteq [-r_0 - \Gamma(t), r_0 + \Gamma(t)] \subset (-a, a)$ **for every time** $t \in [0, T_{speed}]$**. For** $0 < n < 2$ **and** $m \leq n + 2$**, there is a constant** C **such that** $\Gamma(t) \leqslant Ct^{1/(n+4)}$**.**

[Sketch of the proof]

[Stampacchia's lemma] ***Let the nonnegative continuous non-increasing function*** $f(s) : [s_0, \infty) \to \mathbb{R}^1$ ***satisfies the following functional relation:***

$$f(s + f(s)) \leqslant \varepsilon\, f(s)\ \forall\, s \geqslant s_0,\ \ 0 < \varepsilon < 1.$$

Then $f(s) \equiv 0\ \forall\, s \geqslant s_0 + (1 - \varepsilon)^{-1} f(s_0)$**.**

Finite-time blow-up and critical exponents.

Whether or not there is a finite–time singularity, such as $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to T^* < \infty$, is strongly affected by the nonlinearity in the PDE.

$$u_t = u_{xx} + u^p$$

- **if $p \leqslant 1$ then a solution of an initial value problem exists for all time**
- **if $1 < p \leqslant 3$, then any non-trivial solution blows up in finite time**
- **if $p > 3$ then some initial data yield solutions that exist for all time and other initial data result in solutions that have finite–time singularities**

The blow-up is of a focussing type: there are isolated points in space around which the graph of the solution narrows and becomes taller as $t \uparrow T^*$, converging to delta functions centered at the blow–up points.

Scaling argument. $h_t = -(f(h)\,h_{xxx})_x - (g(h)\,h_x)_x$

Consider a solution with a height-scale H and length-scale L. Nonnegativity and volume conservation require that

$$HL \leq V,$$

where V is the total fluid volume. The critical regime should correspond to the balance of nonlinear terms:

$$\frac{f(H)\,H}{L^4} \sim \frac{g(H)\,H}{L^2} \Rightarrow \frac{f(H)}{g(H)} \sim L^2.$$

This suggests that solution can grow without bound only if $\lim_{y\to\infty} \frac{y^2 f(y)}{g(y)} < \infty$**.**

$$\dot{H} \leq \frac{g(H)\,H}{L^2} \sim \frac{g(H)^2}{f(H)}\,H.$$

This suggests that any blow-up must take infinite time whenever $\lim_{y\to\infty} \frac{g(y)^2}{f(y)} = A \leq \infty$ **(dominant e^{At}).**

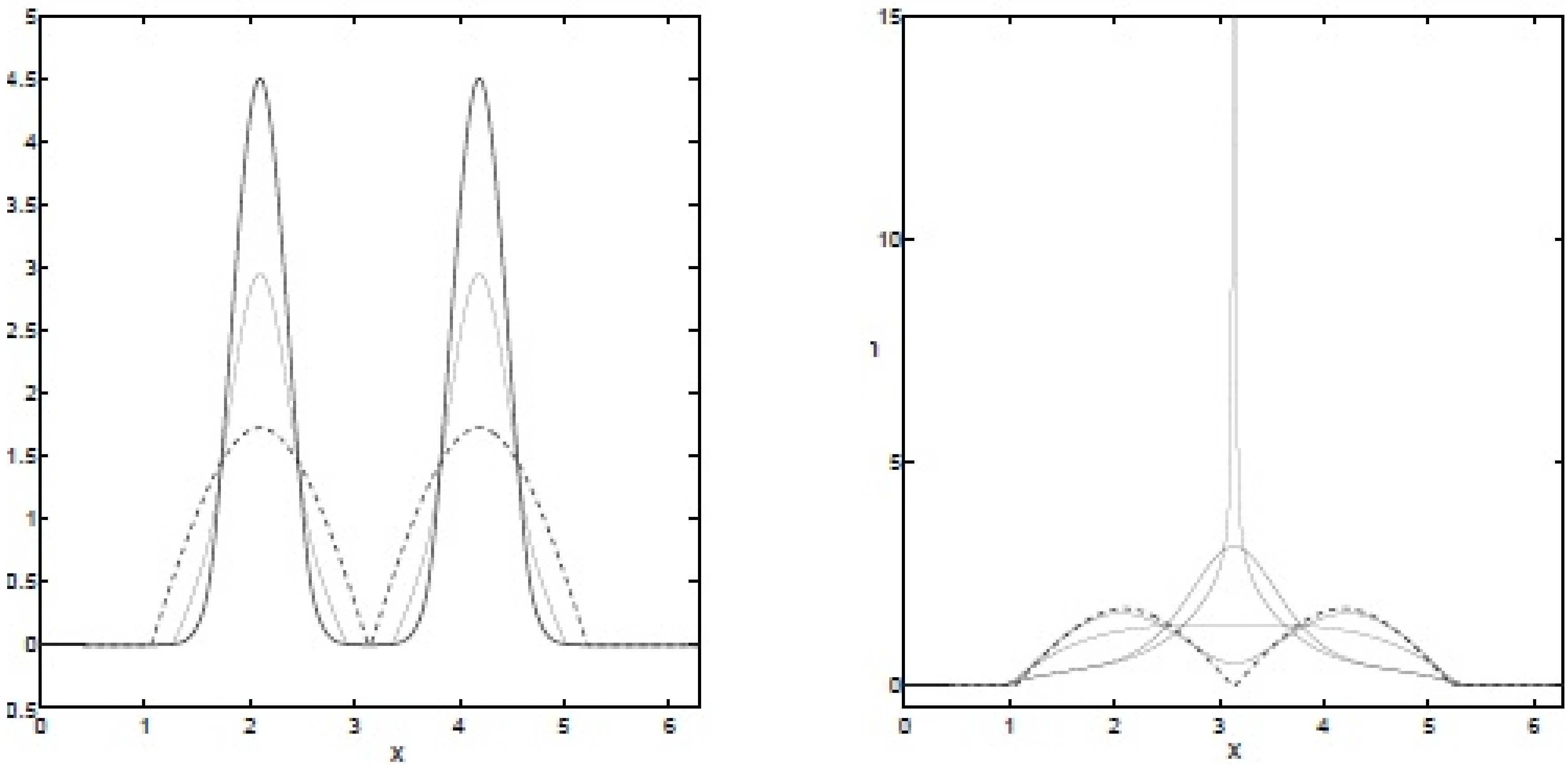
Scaling argument. $h_t = -a_0(h^n h_{xxx})_x - a_1(h^m h_x)_x$

$$\dot{H} \leq \frac{g(H)\, H}{L^2} \sim \frac{g(H)^2}{f(H)} H.$$

This suggests that any blow-up must take infinite time whenever $\lim_{y\to\infty} \frac{g(y)^2}{f(y)} = A \leq \infty$ **(dominant** e^{At}**).**

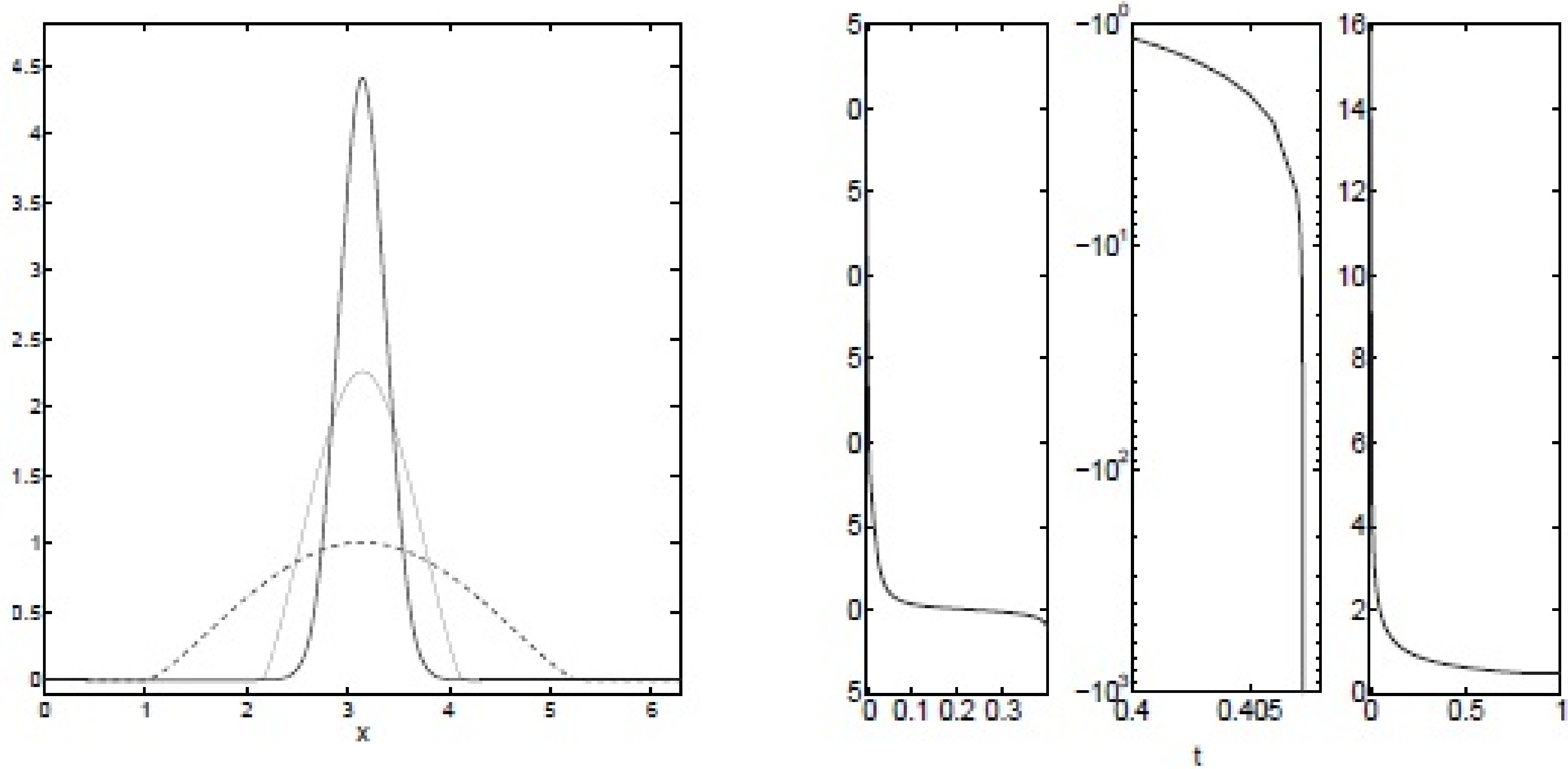
This simple scaling argument suggests that if $0 < n \leq m < n+2$ **then nonnegative solutions are bounded for all time and if** $m > n+2$ **than finite-time blow-up is possible.**

Exponents $n = 2, \ m = 5$.



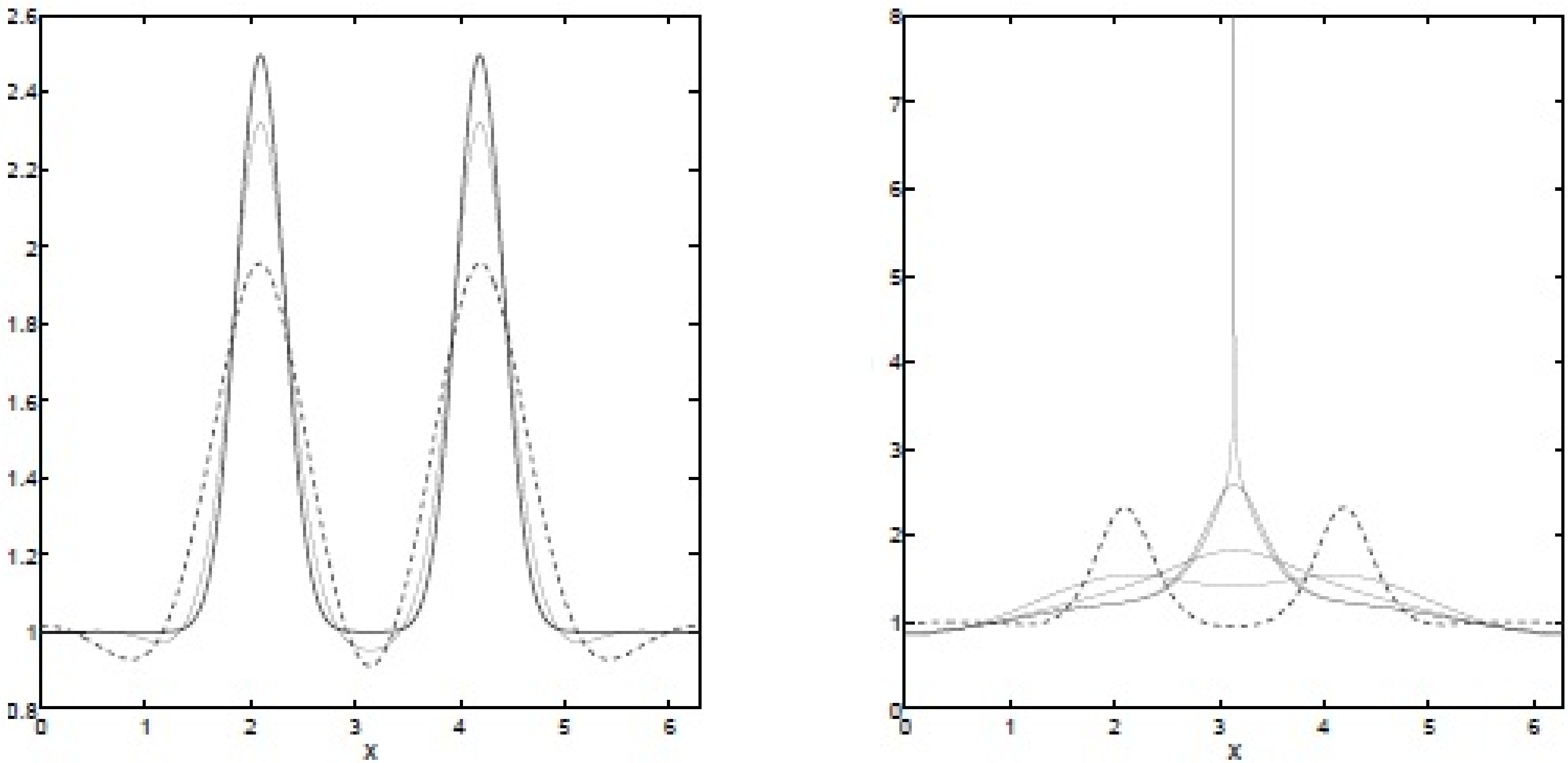
The slow coarsening dynamic and finite speed of the support propagation (left). One-point concentrated blow-up for compactly supported initial data (right).

Exponents $n = 2,\ m = 5$.



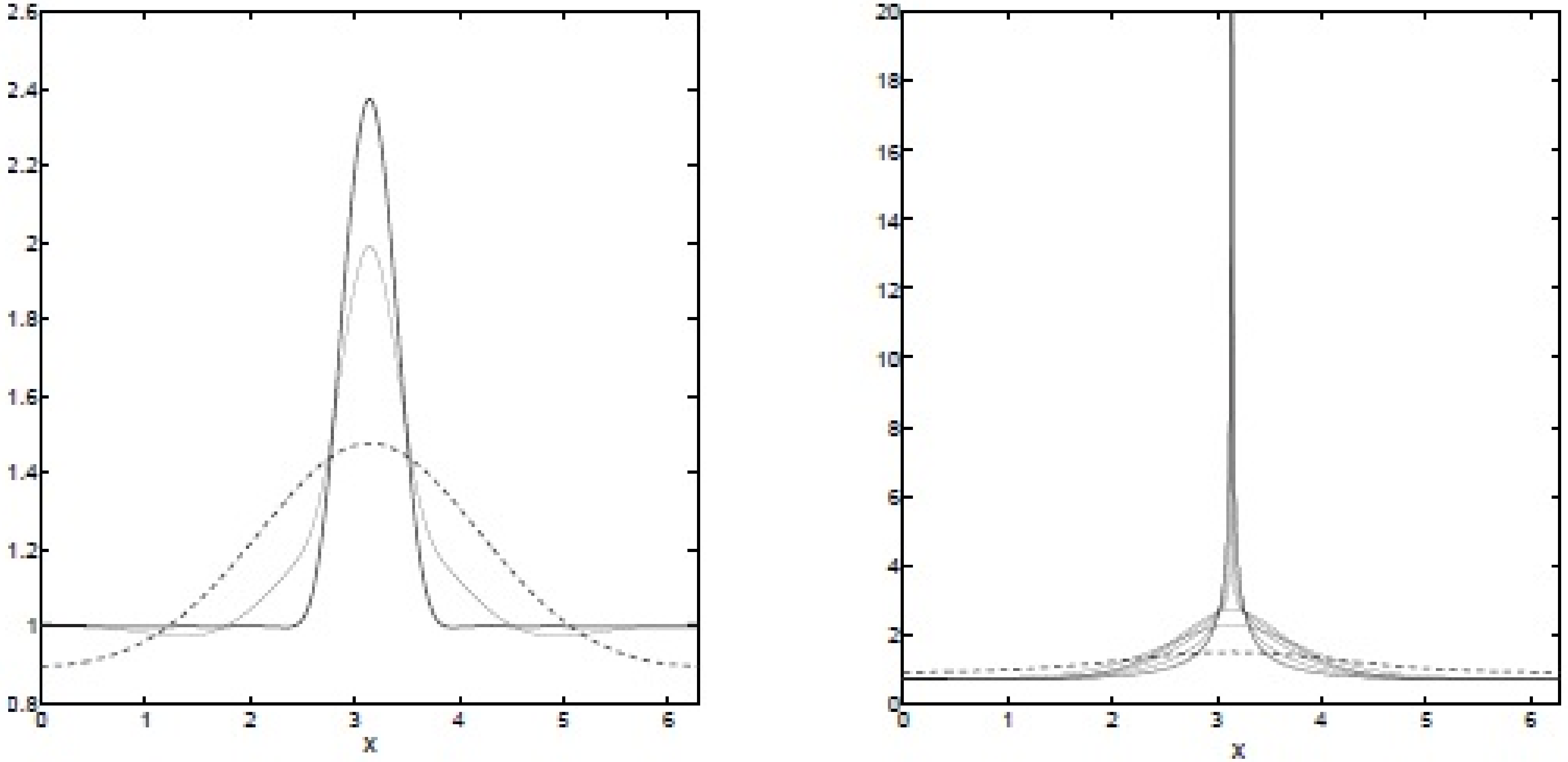
Convergence to a steady state for compactly supported initial data (left). Plots of energy functions (right).

Exponents $n = 2, \ m = 5$.



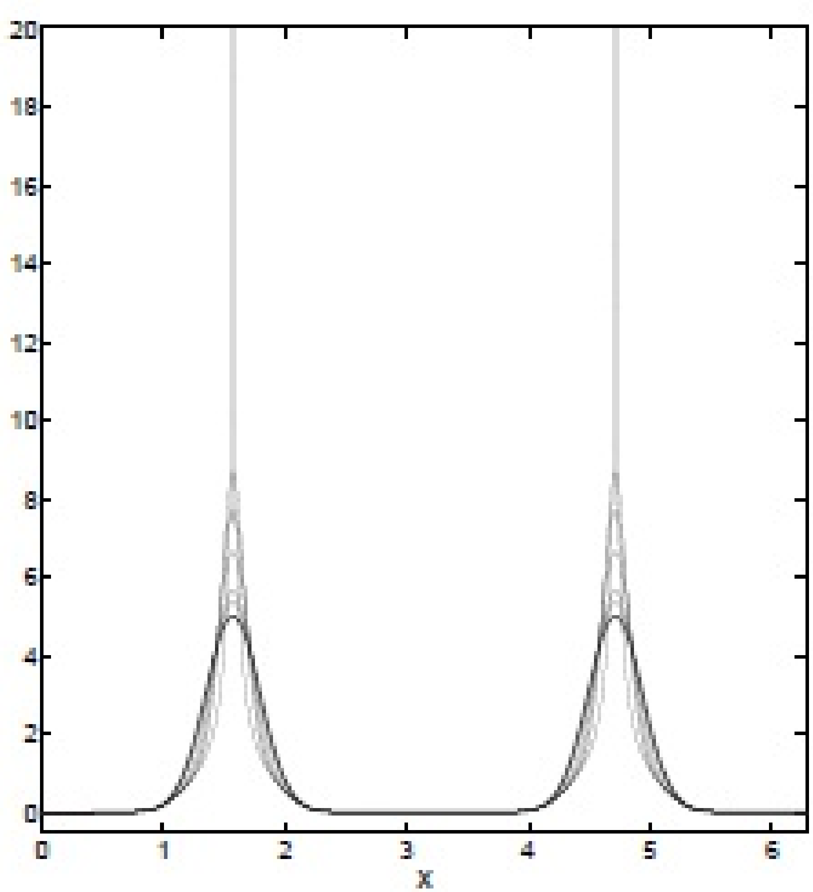
The coarsening dynamic (left) and one-point concentrated blow-up for uniformly positive initial data (right).

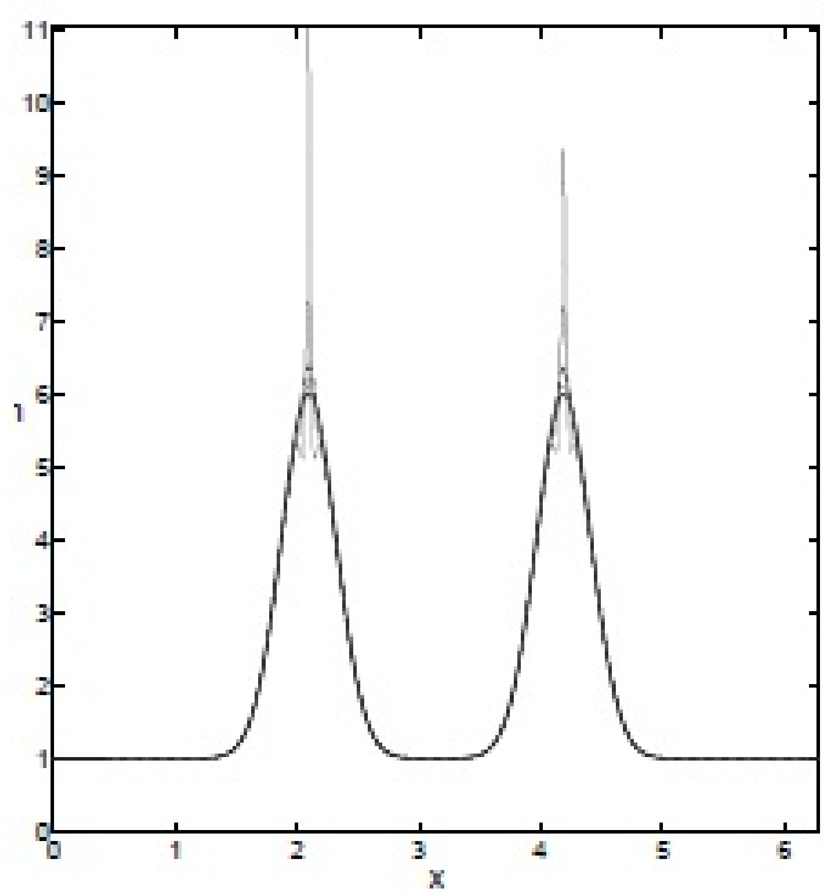
Exponents $n = 4,\ m = 6$.



One-point concentrated blow-up for uniformly positive initial data

Exponents $n = 1.5,\ m = 6$.





Symmetric and non-symmetric two-point concentrated blow-up solutions.

Scaling argument. $h_t = -a_0(h^n h_{xxx})_x - a_1(h^m h_x)_x$

[M. C. Pugh and A. L. Bertozzi, 1999]

First analytical result (for the special case $n = 1$):

Let h_0 be nonnegative and compactly supported, $h_0 \in H^1(R)$.
If $m \geq 3$ and

$$E(0) = \frac{1}{2}\int_{-\infty}^{+\infty} h_{0x}^2(x)\,dx - \frac{1}{m(m+1)}\int_{-\infty}^{+\infty} h_0^{m+1}(x)\,dx < 0,$$

then there is a singular time $T^* < \infty$ and a compactly supported nonnegative weak solution on $[0, T^*)$ such that

$$\limsup_{t\to T^*} ||h(.,t)||_{L^\infty(R)} = \limsup_{t\to T^*} ||h(.,t)||_{H^1(R)} = \infty.$$

Finite time blow-up $n = 1$. $h_t = -(h\, h_{xxx})_x - (h^m\, h_x)_x$

[M. C. Pugh and A. L. Bertozzi, 1999]

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then there is a singular time $T^* < \infty$ and a compactly supported nonnegative weak solution on $[0, T^*)$ such that

$$\limsup_{t \to T^*} ||h(., t)||_{L^\infty(R)} = \limsup_{t \to T^*} ||h(., t)||_{H^1(R)} = \infty.$$

Example of initial values:

$h_0(x) = \lambda(1 + \cos(\lambda\, x))$ **for** $(-\pi/\lambda \leq x \leq \pi/\lambda)$, $\lambda > 0$, $m = 3$,
$E(0) = \frac{-11}{48}\pi\lambda^3$.

Outline of the proof. $h_t = -(h\, h_{xxx})_x - (h^m\, h_x)_x$

- Given $m \geq 3$ and nonnegative periodic initial data h_0 there exists a periodic weak solution on $[-a, a] \times [0, T_0]$ (local in time existence).
- Time T_0 depends on m and $||h_0||_{H^1}$ only.
- Given compactly supported initial data the above solution has finite speed propagation of the support. This speed is controlled by a function of m and $||h_0||_{H^1}$. One can extend the weak solution to the line.
- The solution h can be continued in time if H^1 norm of h is bounded: $(0 < T_0 < T_1 < T_2 ... < T_n < ...)$.
- There is some time T^*, determined by h_0 and m past which this solution can not exist. It then follows that H^1 norm and as a consequence L^∞ norm must have blown up at or before time T^*.

Outline of the proof. $h_t = -(h\, h_{xxx})_x - (h^m\, h_x)_x$

- **The solution h can be continued in time if H^1 norm of h is bounded: $(0 < T_0 < T_1 < T_2 ... < T_n < ...)$.**
- **There is some time T^*, determined by h_0 and m past which this solution can not exist. It then follows that H^1 norm and as a consequence L^∞ norm must have blown up at or before time T^*. Time T^* originates from the second moment inequality.**

$$\int_{-\infty}^{+\infty} x^2\, h(x, T_n)\, dx \le \int_{-\infty}^{+\infty} x^2\, h_0(x)\, dx + 6T_n E(0)$$

$$E(0) = \frac{1}{2} \int_{-\infty}^{+\infty} h_{0x}^2(x)\, dx - \frac{1}{m(m+1)} \int_{-\infty}^{+\infty} h_0^{m+1}(x)\, dx < 0.$$

Let $0 < n < 2$, $m \geqslant \max\{n+2, 4-n\}$**. Then a weak solution** $h(x,t)$ **satisfies the entropy second-moment inequality:**

$$\int\limits_{R^1} x^2 G(h(x,T))\,dx \leqslant e^{B(T)} \Big(\int\limits_{R^1} x^2 G(h_0)\,dx + \int\limits_0^T \Big(k_1 E(0) + k_2 \int\limits_{R^1} x^2 h_{xx}^2\,dx \Big) e^{-B(t)} dt \Big)$$

for all $T \in [0, T_{loc}]$**, where** $k_1 = 2(4-n)$, $k_2 = \frac{3a_0(n-1)}{2}$**. Here**

$$G(z) = \frac{1}{2-n} z^{2-n}, \;\; B(T) := \frac{a_1^2 (1-n)(2-n)}{2a_0(m-n+1)^2} \int\limits_0^T \|h(.,\tau)\|_{L^\infty(R^1)}^{2m-n}\,d\tau.$$

Second moment entropy inequality:

The second-moment inequality can be simplified:

$$\int_{R^1} x^2 G(h(x,T))\,dx \leqslant e^{B(T)} \left(\int_{R^1} x^2 G(h_0)\,dx + k_1\, E(0) \int_0^T e^{-B(t)} dt \right)$$

for all $T \in [0, T_{loc}]$**, where** $k_1 = 2(4-n)$**.**

Here

$$G(z) = \tfrac{1}{2-n} z^{2-n}, \;\; B(T) := \tfrac{a_1^2 (1-n)(2-n)}{2a_0(m-n+1)^2} \int_0^T \|h(.,\tau)\|^{2m-n}_{L^\infty(R^1)}\, d\tau,$$

Introduce: $g(t) := \int_0^t e^{-B(s)} ds$ **by a-priori estimates for** T_i **we obtain the low bound:**

$$g(T_i) \ge C\, T_i.$$

Finite time blow-up:

Let $4 - n \leqslant m < 6 - n$ **with** $0 < n \leqslant \frac{1}{2}$**,**
or $m \geqslant 4 - n$ **with** $\frac{1}{2} < n \leq 1$**,**
or $n + 2 \leq m < 3n + 4$ **with** $1 < n < 2$**.**
Assume that $h_0 \geq 0$**,** $h_0 \in H^1(R^1)$ **and supp** $h_0 \subset (-r_0, r_0)$**, where** $r_0 < a$**. If the energy functional is negative on the initial data** h_0**, then there exists a critical time** T^* **and a compactly supported at any time** $T : 0 < T < T^*$ **generalized weak solution** h **such that**

$$\limsup_{t \to T^*} \|h(.,t)\|_{H^1(R^1)} = \limsup_{t \to T^*} \|h(.,t)\|_{L^\infty(R^1)} = +\infty.$$

Mass concentration property

Bourgain proved a mass concentration property for the solution to cubic NLS ($L^2(R^2)$)

$$iu_t + \triangle u + \lambda |u|^2 u = 0, \quad u_0 \in L^2(R^2)$$

that blows up at a finite time T^*.

The proof was based on the energy equality $E(t) = E_0$ and the result was:

$$\limsup_{t \to T^*} \sup_{I < (T^* - \epsilon)^{1/2}} \left(\int |u|^2 dx \right)^{1/2} > C$$

where C is some universal constant.

We obtained a similar result for the thin-film equation and $\int_\Omega u dx$.

Multidimensional case

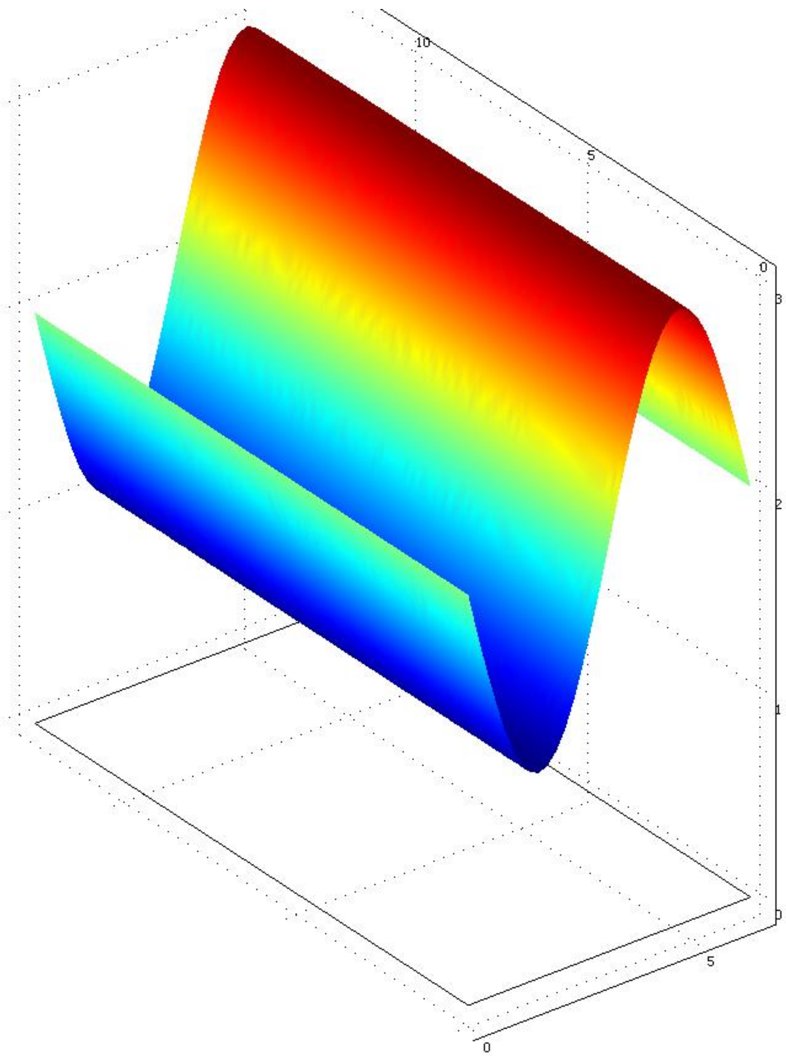
Existence of nonnegative weak and strong solutions for the unstable thin film equation in multi-dimensional domain R^N was recently studied in [J.R. King, R. Taranets, Nonlinear Differ. Eqn. Appl., 2013]

$$h_t + a_0\, div(h^n \nabla \triangle h) + a_1\, div(h^m \nabla h) = 0.$$

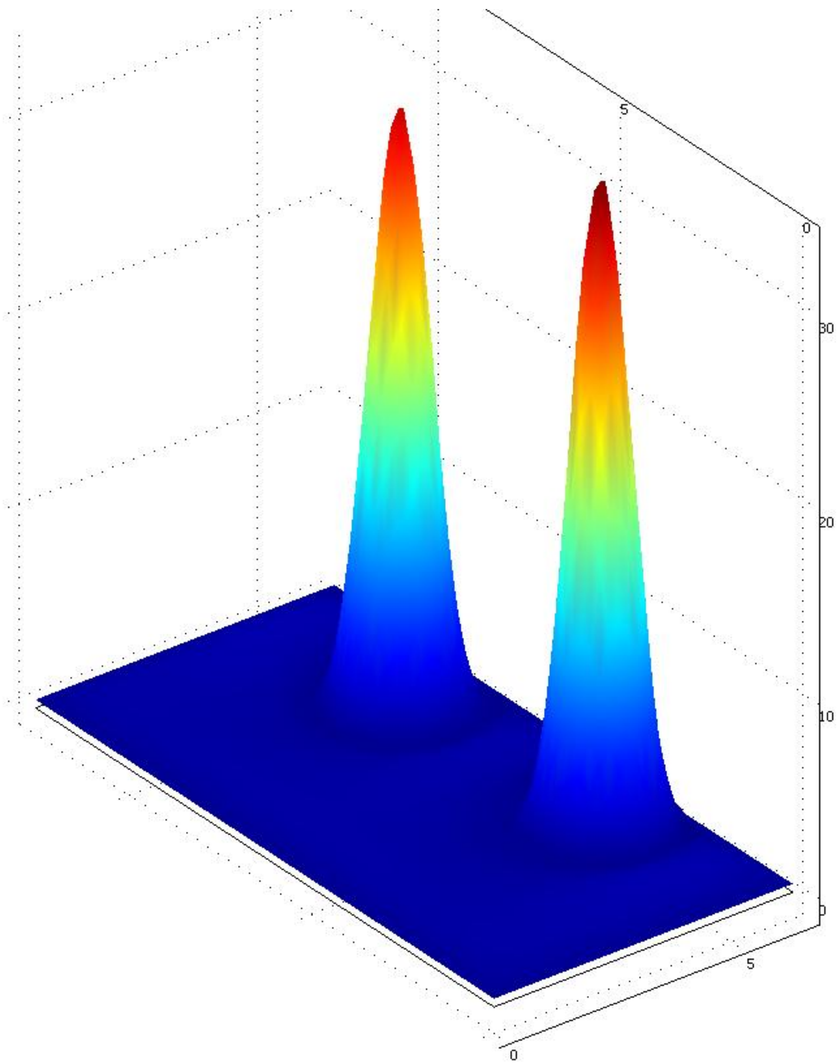
Global existence was shown for $n-2 < m < n+2/N$ **and for** $m = n+2/N$ **under an additional condition that** $M < M_c$**.**

Finite time blow-up was predicted for the case $m > n+2/N$ **and finite time rapture was predicted for the case** $m < n-2$**.**

Multidimensional case (R^2, $n = 2$, $m = 7/2$).



Multidimensional case (R^2, $n = 2$, $m = 7/2$).



Thank you !

THANK YOU FOR YOUR ATTENTION

THE END.