Ref: "The geometry of optimal transportation"

"Five lectures on optimal transportation: geometry, regularity and applications"

"A glimpse into the differential topology and geometry of optimal transport"

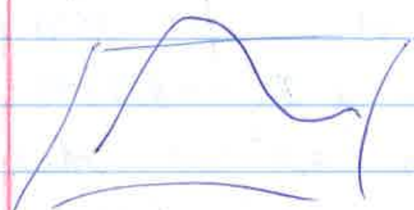
(These three references could be found on McCann's Website)

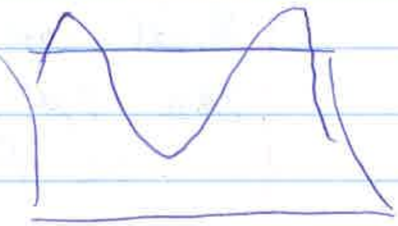
www.math.toronto.edu/mccann/publications

$c(x,y)$ = cost per unit being transported from x to y.

$M^{\pm} \subseteq \mathbb{R}^{n\pm}$, manifolds

$d\mu^{\pm} = f^{\pm}\,dx$





Monge: $F: M^+ \to M^-$, s.t. $F\#\mu^+ = \mu^-$ (i.e. $\mu^+[F^{-1}(V)] = \mu^-[V]\ \forall V \subseteq M^-$)

(Note: if F is a diffeo., then $|\det DF(x)| = \dfrac{f^+(x)}{f^-(F(x))}$)

the problem is: $\displaystyle \inf_{F\#\mu^+=\mu^-} \int_{M^+} c(x, F(x))\, d\mu^+(x)$

Kantorovich (1942): seek $\gamma \in \Gamma = \Gamma(\mu^+,\mu^-) = \left\{ \gamma \geq 0 \text{ on } N = M^+ \times M^- \;\middle|\; \begin{array}{l} \mu^+(U) = \gamma(U\times M^-)\ \forall U \subseteq M^+ \\ \mu^-(V) = \gamma(M^+\times V)\ \forall V \subseteq M^- \end{array} \right\}$

(e.g. if $F\#\mu^+ = \mu^-$, then $\gamma_0 := (id, F)\#\mu^+ \in \Gamma(\mu^+,\mu^-)$)

s.t. $\displaystyle \inf_{\gamma\in\Gamma} cost(\gamma) = \inf_{\gamma\in\Gamma} \int_{M^+\times M^-} c(x,y)\, d\gamma(x,y)$ minimization of a linear function on a convex domain,

$c \in C(\overline{M^+ \times M^-})$, $\Gamma \subseteq C(\overline{M^+ \times M^-})^* = \mathcal{M}(M^+ \times M^-)$

(Γ is weak-* compact in $\mathcal{M}(M^+ \times M^-)$ by Banach-Alaoglu Thm)

- minimizer exists. and Issues:

1). unique ?
2). solve Monge's problem ?
3). Characterization ?
4). further geometric + analytic properties ?

Model case: BRENIER (1987), etc. for $c(x,y) = |x-y|^2$ and $\mu^+ \ll \mathcal{H}^n$

Thm. $\forall \mu^-$, $\exists$ convex function $u : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$,
s.t. $F(x) := Du(x)$. satisfies $F \# \mu^+ = \mu^-$ that
is unique μ^+-a.e. and uniquely solves
Monge's and Kantorovich problem

Regularity of F for $c(x,y) = |x-y|^2/2$.

Delanoë $(n=2)$, Caffarelli (91-96) Urbas (1997)

Thm If $d\mu^\pm = f^\pm \, dx$, $M^\pm \subseteq \mathbb{R}^n$, M^- convex. $\log f^\pm \in L^\infty(M^\pm)(\cap C^\infty)$
then $F = Du \in C^\alpha (\cap C^\infty)$ for some $\alpha > 0$.

for general cost, Ma - Trudinger - Wang

$L^\infty \Rightarrow C^\alpha$ Figalli - Kim - McCann.

γ is extremal in Γ unless it's the midpoint of a segment in Γ.
e.g. $(id, F)\#\mu^+$ is extremal in Γ, but not all extremal points have this form.

$spt\,\gamma = S \subseteq N := M^+ \times M^-$

smallest closed set in N, carrying full mass for γ.

If $c \in C^2$ near (x_0, y_0), then its local topology determine dimension of spt γ nearby. ~~[illegible]~~

$$\delta(x,y;x_0,y_0) := -c(x,y) - c(x_0,y_0) + c(x,y_0) + c(x_0,y) \text{ on } N = M^+ \times M^-$$

Observe: if γ is optimal, then $\delta \geq 0$ on S^2, where $S = spt\,\gamma$.

Fix $(x_0, y_0) \in S = spt\,\gamma$

Set $\delta_0(x,y) = \delta(x,y;x_0,y_0)$.

Note $D_{(x,y)}\delta_0|_{(x_0,y_0)} = 0$

Define $h_{x_0y_0} = Hess_{(x,y)}\,\delta_0|_{(x_0,y_0)}$.

Taylor expansion: ~~$\delta_0(x+\Delta x, y) = \delta(x,y) +$~~

$$\delta_0(x_0+\Delta x, y_0+\Delta y) = \delta_0(x_0,y_0) + \frac{1}{2}(\Delta x, \Delta y)\, h_{x_0y_0}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} + l.o.t$$

Symmetry

$sig(h) = (h_+, h_0, h_-)$, $h_0 + h_+ + h_- = n_+ + n_-$ $\quad h = \left(\begin{array}{c|c} 0 & D_{xy}c \\ \hline (D_{yx}c)^T & 0 \end{array}\right)$

if $h\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \lambda\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}$, $\lambda > 0$; $\Rightarrow$ ~~$h_+ = h_-$~~

$h_+ = h_- = \frac{n_- + n_+ - h_0}{2}$

$\therefore h_+ + h_0 = \frac{n_+ + n_- + h_0}{2}$

$\Rightarrow h\begin{pmatrix}\Delta x\\ -\Delta y\end{pmatrix} = -\lambda\begin{pmatrix}\Delta x\\ -\Delta y\end{pmatrix}$

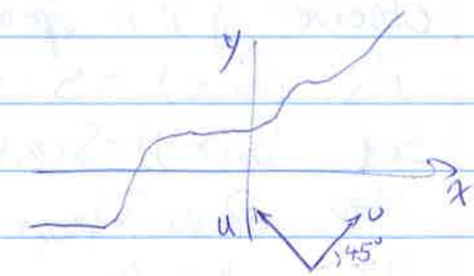
e.g. $\left.\begin{array}{l} n_+ = n_- \\ h_0 = 0 \end{array}\right\} \Rightarrow$ ~~$h_+ + h_0$~~ $h_+ + h_- = 2n_+$

<u>Thm</u> (Pass, McCann-Pass-Warren)

Suppose $c \in C^2(M^+ \times M^-)$, $\mu^\pm$ compactly supported; γ be a solution of Kantorovich problem. Suppose $(x_0, y_0) \in spt\,\gamma$ and c is non-degenerate at (x_0, y_0). <u>Then</u> there is a neighbourhood N of (x_0, y_0), such that $N \cap spt\,\gamma$ is contained in an n-dimensional Lipschitz submanifold. In particular, if $D^2_{xy} c$ is nonsingular everywhere, $spt\,\gamma$ is contained in an n-dimensional Lipschitz submanifold.

Sketch of proof

$$\frac{\partial^2 c}{\partial x \partial y} < 0 \Rightarrow$$



Does Monge's problem have a solution? ($\mu \ll H^n$)

by Gangbo (1996), Levin (1999) Yes if

$(A_1)_+$ $\begin{cases} c \in C^1(\bar{N}).\ \forall x_0 \in M^+,\ \forall y \neq y_0 \in M^-, \text{ the map } x \in M^+ \to S_0(x,y) \\ \text{has no critical pts.} \end{cases}$

<u>Then</u> $spt\,\gamma \subseteq Graph(F)$ for some $F: M^+ \to M^-$

e.g. $c(x,y) = |x-y|^p$, $p > 1$, on $M^\pm = \mathbb{R}^n$, for $p = 2$, $S_0(x,y) = 2(x-x_0)(y-y_0)$

Notice $(A_1)_+$ can't be satisfied if M^+ is compact

possible solution: relax C^1 hypothesis e.g. $c = d^2$ on any Riemann manifold ($M^\pm = (M, g)$) $\exists$ Monge solution & unique

(McCann 2001)

Thm (Chiappori – McCann – Nesheim)

$c \in C^1$, $\mu^+ << dx$, if $\forall x_0 \in M^+$, $y \neq y_0 \in M^-$, $x \in M^+ \longrightarrow S_0(x,y)$ has at most 2 critical points (a global min & max) then the Kantorovich solution is unique (but may not be Monge)

open question: can such a condition exist $M^+ = \mathbb{T}^2$ or other topology pi?

Regularity for general cost requires

(A0) $c \in C^4(\bar{N})$ $\forall (x_0, y_0) \in N$.

(A1) $y \in M^- \longmapsto D_x c(x_0, y)$ and $x \in M^+ \longmapsto D_y c(x, y_0)$ are injective.

(A2) $\det D^2_{x^i y^j} c(x_0, y_0) = \det(c_{ij}) \neq 0$.

(A3) $\mathrm{cross}(p,q) \geq 0$ for all $(p,q) \in T_{(x_0,y_0)} M^+ \times M^-$ such that $p^i c_{i,j} q^j = 0$

(A4) $M^-_{x_0} := D_x c(x_0, M^-) \subseteq \mathbb{R}^n$ and $M^+_{y_0} := D_y c(M^+, y_0) \subseteq T^*_{y_0} M^-$ are convex

where $\mathrm{cross}(p,q) = \mathrm{sec}^{(N,h)}_{(x_0,y_0)} \quad p \oplus 0 \wedge 0 \oplus q$

depends on the secondary curvature of the semi Riemannian metric h on $N = M^+ \times M^-$