OPERADS IN ALGEBRAIC TOPOLOGY I

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Contents

Introduction

I'm really going to start from the beginning and build things up because operads are really important tools in algebraic topology.

Slogan. Operads encode n-ary operations and relations among them.

Motivating examples.

(i) Associative monoids. Suppose I have a set X with a binary multiplication $\mu\colon X \times X \to X$ such that μ is associative:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu \times X} & X \times X \\ {\scriptstyle X\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ X \times X & \xrightarrow[\mu]{} & X \end{array}$$

From the binary operation (X, μ) we get higher n-ary operations $X^{\times n} \to X$. In fact there exist $n!$ distinct n-ary operations, depending on how one permutes the inputs.

(ii) Commutative monoids. Suppose we have (X, μ) as before where μ is associative and commutative: $\mu(x_1, x_2) = \mu(x_2, x_1)$. Again, this generates higher n-ary operations $X^{\times n} \to X$ but now there is a unique n-ary operation for each n.

(iii) Based loop spaces. Let X be a based space and consider the space ΩX of based loops. We can define $\mu\colon \Omega X \times \Omega X \to \Omega X$ that sends a pair of loops to their concatenation: $(\lambda_1, \lambda_2) \mapsto \lambda_1 * \lambda_2$. This operation isn't strictly associative, but it is homotopy associative. There exists a based homotopy $H\colon \mu(\mu \times \Omega X) \simeq_* \mu(\Omega X \times \mu)$. Moreover, there exists a homotopy between the two homotopies from

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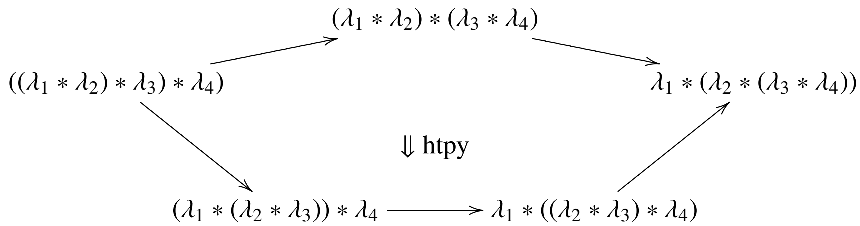
$\mu(\mu(\mu \times \Omega X \times \Omega X))$ to $\mu(\Omega X \times \mu(\Omega X \times \Omega X \times \mu))$ induced by H.



(iv) Double loop spaces: $\Omega^2 X$. This has two homotopy associative multiplications that satisfy an up-to-homotopy Eckmann-Hilton relation. This implies that they are homotopy commutative and that they are the same up to homotopy.

Operads: definitions and elementary examples

Let $(\mathcal{M}, \otimes, I)$ be a closed symmetric monoidal category. Here *closed* means that $-\otimes X\colon \mathcal{M} \to \mathcal{M}$ has a right adjoint $\mathrm{Hom}(X, -)\colon \mathcal{M} \to \mathcal{M}$. In particular, the counit of this adjunction defines a natural morphism

$$\mathrm{ev}_X\colon \mathrm{Hom}(X, Y) \otimes X \to Y$$

for all $X, Y \in \mathcal{M}$. I also want $\mathcal{M}$ to be cocomplete.

Example. $(\mathbf{sSet}, \times, \{*\})$, $(\mathbf{Top}^{\mathrm{nice}}, \times, \{*\})$, $(\mathbf{Ch}_R, \otimes, R)$.

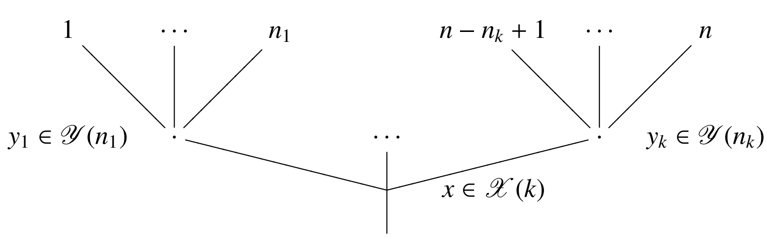
Definition. Let $\mathcal{M}^\Sigma$ denote the category of *symmetric sequences* in $\mathcal{M}$. Objects are $\mathscr{X} = (\mathscr{X}(n))_{n\geq 0}$ where $\mathscr{X}(n) \in \mathcal{M}$ is an object that admits a Σ_n-action (i.e., there exists a homomorphism $\Sigma_n \to \mathrm{Aut}(\mathscr{X}(n))$). A morphism $f\colon \mathscr{X} \to \mathscr{Y}$ is a collection of Σ_n-equivariant maps $f_n\colon \mathscr{X}(n) \to \mathscr{Y}(n)$ for all $n \geq 0$.

Remark. $\mathcal{M}^\Sigma$ admits several monoidal structures. For example,

- the *level monoidal structure* $\otimes$ with $(\mathscr{X} \otimes \mathscr{Y})(n) = \mathscr{X}(n) \otimes \mathscr{Y}(n)$ given the diagonal Σ_n-action. The unit is $\mathscr{C} = (\mathscr{C}(n))$, where $\mathscr{C}(n) = I$.
- the graded monoidal structure, the matrix monoidal structure
- the *composition monoidal structure* $\circ$, which is non-symmetric. For $n \geq 1$, define

$$(\mathscr{X} \circ \mathscr{Y})(n) = \coprod_{k\geq 1, \vec{n}=(n_1,\ldots,n_k), \sum n_i = n} \mathscr{X}(k) \otimes_{\Sigma_k} (\mathscr{Y}(n_1) \otimes \cdots \otimes \mathscr{Y}(n_k)) \otimes_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}} I[\Sigma_n]$$

where $I[\Sigma_n]$ is the tensor of $n!$ copies of the monoidal unit I. The unit for $\circ$ is the symmetric sequence $\mathscr{I}$ that has I in arity one and the initial object everywhere else. Here is a schematic picture:



Recall. A *monoid* in a monoidal category $(\mathcal{M}, \otimes, I)$ is $A \in \mathcal{M}$, a multiplication $\mu\colon A\otimes A \to A$, and a unit $\eta\colon I \to A$ so that

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mu\otimes A} & A\otimes A \\ {\scriptstyle A\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\otimes A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} A\cong A\otimes I & \xrightarrow{A\otimes\eta} & A\otimes A & \xleftarrow{\eta\otimes A} & I\otimes A\cong A \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle =} & \\ & & A & & \end{array}$$

Definition. An *operad* in $\mathcal{M}$ is a monoid in $\mathcal{M}^{\Sigma}$ with respect to the composition monoidal structure. I.e., $(\mathscr{P}, \mu, \eta)$ where $\mathscr{P} = (\mathscr{P}(n))_{n\geq 0}$, $\mu\colon \mathscr{P}\circ\mathscr{P} \to \mathscr{P}$, $\eta\colon \mathscr{I} \to \mathscr{P}$. Here

$$\mu \leftrightsquigarrow \{\mu_{k,\vec{n}}\colon \mathscr{P}(k)\otimes(\mathscr{P}(n_1)(\otimes\cdots\otimes\mathscr{P}(n_k)) \to \mathscr{P}(n))\}_n$$

with appropriate equivariance conditions.

Example.

(i) Let $X \in \mathcal{M}$. The *endomorphism operad* on X is $\mathrm{End}(X)$ where $\mathrm{End}(X)(n) = \mathrm{Hom}(X^{\otimes n}, X)$ and $\mu\colon \mathrm{End}(X)\circ\mathrm{End}(X) \to \mathrm{End}(X)$ corresponds to

$$\mu_{k,\vec{n}}\colon \mathrm{Hom}(X^{\otimes k}, X)\otimes(\mathrm{Hom}(X^{\otimes n_1}, X)\otimes\cdots\otimes\mathrm{Hom}(X^{\otimes n_k}, X)) \to \mathrm{Hom}(X^{\otimes n}, X)$$

given by $f\otimes(g_1\otimes\cdots g_k) \mapsto f\circ(g_1\otimes\cdots\otimes g_k)$. (This can also be defined abstractly using the adjunctions.)

(ii) The *associative operad* $\mathrm{As}(n) = I[\Sigma_n]$. Here

$$\Sigma_k\times(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}) \xrightarrow{\mu} \Sigma_n$$

is defined so that $\sigma(\tau, \tau_1, \ldots, \tau_k)$ is the permutation of n letters partitioned into k boxes where each τ_i permutes the elements of a box and τ permutes the boxes.

(iii) The *commutative operad* is $\mathrm{Com} = \mathscr{C}$ with $\mathscr{C}(n) = I$. Here μ consists of the unit isomorphisms $I\otimes(I\otimes\cdots\otimes I) \xrightarrow{\cong} I$. This was the monoidal unit for the level monoidal structure.

Algebras

Definition. Let $(\mathscr{P}, \mu, \eta)$ be an operad in $\mathcal{M}$. A *$\mathscr{P}$-algebra* is an object $X \in \mathcal{M}$ together with an operad map $\phi\colon \mathscr{P} \to \mathrm{End}(X)$.

You can think of this as being a representation of $\mathscr{P}$ on X. What does this mean? The components of the map are Σ_n-equivariant maps

$$\phi_n\colon \mathscr{P}(n) \to \mathrm{End}(X)(n) = \mathrm{Hom}(X^{\otimes n}, X).$$

These transpose to maps

$$\phi_n^\flat\colon \mathscr{P}(n)\otimes_{\Sigma_n} X^{\otimes n} \to X.$$

Since ϕ is an operad map, there exist an "associativity" relation:

$$\begin{array}{ccccc} & & \mathscr{P}(k)\otimes(\mathscr{P}(n_1)\otimes\cdots\mathscr{P}(n_k))\otimes X^{\otimes n} & & \\ & {\scriptstyle \mu\otimes X^{\otimes n}}\swarrow & & \searrow{\scriptstyle \mathscr{P}(k)\otimes\phi^\flat_{n_1}\otimes\cdots\otimes\phi^\flat_{n_k}} & \\ \mathscr{P}(n)\otimes X^{\otimes n} & & & & \mathscr{P}(k)\otimes X^{\otimes k} \\ & {\scriptstyle \phi^\flat_n}\searrow & & \swarrow{\scriptstyle \phi^\flat_k} & \\ & & X & & \end{array}$$

Slogan. $\mathscr{P}(n)$ parametrizes the n-ary operations on the $\mathscr{P}$-algebra X.

Example.

(i) An As-algebra is a monoid
$$\mathrm{As}(n) \otimes_{\Sigma_n} X^{\otimes n} \cong X^{\otimes n} \to X.$$

(ii) A Com-algebra is a commutative monoid
$$\mathrm{Com}(n) \otimes_{\Sigma_n} X^{\otimes n} \cong X^{\otimes n}/_{\Sigma_n} \to X.$$

Remark. A morphism of operads $\phi\colon \mathscr{P} \to \mathscr{Q}$ induces a functor $\phi^*\colon \mathbf{Alg}_{\mathscr{Q}} \to \mathbf{Alg}_{\mathscr{P}}$ by pullback of structure: $\mathscr{P} \to \mathscr{Q} \to \mathrm{End}(X)$.

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