SULLIVAN'S CONJECTURE AND APPLICATIONS TO ARITHMETIC

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Abstract. One can phrase interesting objects in terms of fixed points of group actions. For example, class numbers of quadratic extensions of Q can be expressed with fixed points of actions on modular curves. Derived functors are frequently better behaved than their non-derived versions, so it is useful to consider the associated derived functor, called the homotopy fixed points. Sullivan's conjecture is an equivalence between appropriately completed spaces of fixed points and homotopy fixed points for finite p-groups. It was proven independently by H. Miller, G. Carlsson, and J. Lannes. This talk will present Sullivan's conjecture and its solutions, and discuss analogues for absolute Galois groups conjectured by Grothendieck.

This was a chalk talk.

Let G be a finite group, X some set, abelian group, or topological space on which G acts.

$X^G = \{x \mid gx = x \forall\ g \in G\}$ is the set of fixed points.

Example: let N be a positive integer. Define $X_0(N)$ to be the moduli space of (E, ϕ) where E is a (generalized) ellliptic curve and $\phi : E \to E'$ is an isogeny of degree N. A group action of $\mathbb{Z}/2 = \langle \omega_N \rangle$ is given by the Atkin-Lehner involution ω_N, where $\omega_N(E, \phi) = (E/\ker(\phi), E/\ker(\phi) \to E/E[N])$ where $E[N]$ is the N-torsion of E. That map is degree N because you've already divided out by N and $N^2/N = N$. The following is a consequence of work of Ogg:

Theorem 0.1. *Fix $N > 4$. Then*

$$|X^G| = \begin{cases} h(-N) + h(-4N) & \text{if } N \equiv 3 \mod 4 \\ h(-4N) & \text{otherwise} \end{cases}$$

where $h(-N)$ is the order of the class group of $\mathbb{Q}(\sqrt{-N})$

The class group equals the narrow class group for quadratic imaginary extensions. Now define $h(N)$ to be the order of the narrow class group of $\mathbb{Q}[\sqrt{N}]$. This doesn't affect the statement of the theorem above.

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Proof Sketch. Suppose $(E, \phi) \in X_0(N)^{\omega_N}$. Elliptic curves correspond to lattices in C so $H_1(E, \mathbb{Z}) = \Lambda \subset C$.

(E, ϕ) corresponds to the pair (Λ, Λ_2) where Λ is the lattice for E, and the kernel of ϕ determines a larger lattice Λ_2 where $\Lambda \subset \Lambda_2$ has index N.

ω_N takes this pair to $(\Lambda_2, 1/N\Lambda) = (a\Lambda, a\Lambda_2)$ where $1/N\Lambda_1 = a\Lambda_2 = a^2\Lambda_1$ implies $a^2N \in \{\pm 1, \pm i, \pm\zeta_3, \pm\zeta_3^2\}$.

Now, $\mathbb{Z}[a]$ is a lattice in C so the only b for which $b = a^2N$ is $b = -1$ (argue by contradiction, using the fact that extensions can't be too big).

This implies $H_1(E, \mathbb{Z})$ is a module over $\mathbb{Z}[\sqrt{-N}]$.

□

This gives us some sense of why we might expect to see class groups arising in the study of $X_0(N)$.

There is an action of $\mathrm{Gal}(C/\mathbb{R})$ on $X_0(N)/\omega_N$, thus we may consider $(X_0(N)/\omega_N)(\mathbb{R})$-points in $X_0(N)$.

Theorem 0.2 (Another Theorem of Ogg)**.** $\pi_0(X_0(N)/\omega_N(\mathbb{R})) = \frac{1}{2}(h(4N) + 1)$

when N is 2 or 3 mod 4 and square free.

Note that $h(-N)$ gets switched $h(N)$. This has to do with solving $\omega_N(E, \phi) = (\overline{E}, \overline{\phi})$.

There is a conjecture of Gauss from 1801 that there are infinitely many $\mathbb{Q}(\sqrt{N})$ with class number 1. This is still open.

1. Sullivan Conjecture

In the previous section, fixed points were used. Now we'll use homotopy fixed points. Let EG be a contractible space with a free G-action. Then $\mathrm{map}(EG, X)$ has an action of G given by $(g \cdot f)(x) = gfg^{-1}(x)$ for all $x \in EG$.

The *homotopy fixed points* X^{hG} of X are the fixed points $\mathrm{map}(EG, X)^G$. By analogy, $\mathrm{map}(*, X)^G = X^G$. The G-map $EG \to *$ induces a map $X^G \to X^{hG}$ via mapping spaces.

The *homotopy fixed point spectral sequence* is $H^i(G, \pi_j X) \Rightarrow \pi_{j-i}X^{hG}$. It's really a bunch of spectral sequences in one because of different path components. The differential d^r goes left r and up $r - 1$.

Example: $X = S^2$, $\mathbb{Z}/2$ acts on S^2 via $(x, y, z) \mapsto (x, y, -z)$.

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$\pi_n(S^2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
$\mathbb{Z}/2 - act$	$(-) * (-1)$	???				

The unknown action can be worked out via Hilton-Milnor on the Hopf element. You push forward the 'multiplication by -1' action and see that $a_*\eta = a\eta + \binom{a}{2}2\eta = a\eta + a(a-1)\eta = a^2\eta$. Thus, the action is given by multiplication by $(-1)^2 = 1$. The action for $n = 4$ is also trivial.

Denote by $\mathbb{Z}(1)$ the $\mathbb{Z}$ in $n = 2$, and by $\mathbb{Z}(2)$ the $\mathbb{Z}$ in $n = 3$. Then $H^i(\mathbb{Z}/2, \mathbb{Z}(1))$ is $\mathbb{Z}/2$ for i odd and 0 otherwise, while $H^i(\mathbb{Z}/2, \mathbb{Z}(2))$ is $\mathbb{Z}/2$ for i even and positive, and 0 otherwise. Finally, $H^1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$. So the E_2-page is a bunch of $\mathbb{Z}/2$'s convergint to a filtration of $\pi_* X^{hG}$. Meanwhile $X^G = S^1$.

Sullivan's Conjecture (proven by Gunnar Carlsson, Haynes Miller, and Jean Lannes all independently) is saying the homotopy fixed points are not far off from the fixed points. In this case it's saying that the homotopy fixed point spectral sequence is converging to the 2-completion, i.e. the 2-adics $\mathbb{Z}_2 = \pi_1 X^{hG}$. You are filtering $\mathbb{Z}_2 = \pi_1 X^{hG}$ so that successive filtration quotients are $\mathbb{Z}/2$ and this is what is showing up in this spectral sequence.

This is all motivation for the Bousfield-Kan Spectral Sequence.

The punchline is that if X is simply connected then $X \to X_p^\wedge$ (the p-completion of X) is terminal among $H_*(-, \mathbb{F}_p)$-equivalences. This is saying it's a *localization map*.

2. Sullivan's Conjecture

Theorem 2.1 (Sullivan's Conjecture, proven by Miller, Carlsson, Lannes)**.** *Let G be a finite p-group, X a finite G CW-complex, $(X^G)_p^\wedge \to (X_p^\wedge)^{hG}$ is an equivalence.*

Example: let X be an algebraic curve over $\mathbb{R}$ whose Euler-characteristic $\chi(X(C)) < 0$, let $b \in X(\mathbb{R})$, then $\pi_0(X(\mathbb{R})) \cong \pi_0 X(C)^{hGal(C/\mathbb{R})} \cong H^1(\mathrm{Gal}(C/\mathbb{R}), \pi_1 X(C))$.

Here $\pi_1 X(\mathbb{C})$ need not be abelian and the action may not be trivial.

Non-example: $X = S^n$ with a trivial $\mathbb{Z}/2$ action. Then $\mathrm{Sym}^\infty(X) = K(\mathbb{Z}, n)$ with a trivial action. So $(\mathrm{Sym}^\infty X)^G = K(\mathbb{Z}, n)$ but the homotopy fixed points are very different. We may compute them via a spectral sequence:

$H^1(\mathbb{Z}/2, \mathbb{Z}) \Rightarrow \pi_{n-1}(\mathrm{Sym}^\infty X)^{hG}$

This demonstrates that $\pi_{n-i}(\mathrm{Sym}^\infty(X)^{hG}) \cong H^i(\mathbb{Z}/2, \mathbb{Z})$ is non-trivial for many i.

Another non-example is $G = \mathbb{Z}$, $X = \mathbb{R}$, and G acting by translation. Then $X^{hG} \simeq *$.

The main result of this talk (answering a recent question of Haynes Miller) follows. Let $k/\mathbb{Q}$ finite field extension and X a proper smooth curve. So by Falting's Theorem $X(k)$ is a finite set.

Proposition 2.2. *The section conjecture is equivalent to the statement that* $X(k) \xrightarrow{\sim} EtX_{\overline{k}}^{hGal(\overline{k}/k)}$.

This is an enrichment of the section conjecture, which is a statement about π_0.

Proof goes by reducing to an abelian case then doing a spectral sequence computation of the sort discussed earlier.