A K(Z,4) IN NATURE

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Abstract. Topological spaces and simplicial sets have associated homotopy types (that is, the 1-category of topological spaces maps to the infinity-category of spaces, and so does the 1-category of simplicial sets), but these are not the only kind of mathematical objects that have associated homotopy types. In this talk, I will present a mathematical object (not a topological space) that comes from the theory of von Neumann algebras, and whose associated homotopy type is K(Z,4).

This was a chalk talk. The scanned lecture notes can be found at the end.

Joint with A. Bartels, C. Douglas.

1. K(Z,0)-K(Z,3)

$K(\mathbb{Z},0)=\mathbb{Z}$, $K(\mathbb{Z},1)=S^1$ or $\mathbb{C}^x$

$K(\mathbb{Z},2)=\mathbb{CP}^\infty$ the set of all lines in $\mathbb{C}^\infty$. We'd like this better if we keep the description as 'set of all lines' but throw away the reliance on $\mathbb{C}^\infty$. So we consider the stack of all lines. It does have a homotopy type and it's a $K(\mathbb{Z},2)$.

This stack of all lines is the classifying stack of S^1 (if you equip your lines with a metric; otherwise it's $B(\mathbb{C}^x)$).

We can do the same trick to get to $K(\mathbb{Z},3)$ if you have a group which is a $K(\mathbb{Z},2)$. We'll start with an infinite dimensional Hilbert space H and take the unitary group $U(H)$. A theorem of Kuiper says $U(H)$ is contractible. Also, multiples of the unit gives a subgroup of $U(H)$ which is an S^1. When we take the quotient we get $PU(H)=U(H)/S^1$ and this is a $K(\mathbb{Z},2)$. Taking classifying stack $BPU(H)$ yields a $K(\mathbb{Z},3)$.

Aside: The correct topology on $U(H)$ is the one that makes $U(H)$ into a Polish group, i.e. completely metrizable and separable. With this topology, all the topologies on H become equal upon passage to $U(H)$. Furthermore, Kuiper's Theorem is easy to prove. Just take any $u \in U(L^2[0,1])$ and define $u_t : 1 \mapsto u$ so that we obtain a splitting $L^2[0,t] \oplus L^2[t,1]$ where u acts on the first part and 1 on the second. Let t go to 0 and this provides a continuous homotopy where the first part is contracted.

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Note that because $PU(H)$ is defined as a quotient group, it's not clear how to make it act on something. Thankfully,

Theorem 1.1. *$PU(H) = Aut(b(H))$ where b is for 'bounded operators' and Aut means automorphisms respecting the C^*-structure.*

Proof. The action of $PU(H)$ on $b(H)$ is by conjugation.

To see that it is in fact the full automorphism group, first prove that every automorphism of $b(H)$ is inner, using the analytic Morita equivalence $b(H) \simeq_M \mathbb{C}$ (algebraically, $\mathbb{C}$ is only Morita equivalent to $b(V)$ for finite dimensional V). Given an automorphism α, consider the $(b(H), \mathbb{C})$-bimodule H. One could also twist the $b(H)$ action by α and there's a unitary isomorphism U between these two. Thus, $\alpha = ad(U)$.

Having proven surjectivity, we must now prove $\ker(ad : U(H) \to Autb(H)) = S^1$. This kernel is the center $Z(U(H))$. Next, $Z(b(H)) = End({}_{b(H)}b(H)_{b(H)}) = End({}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}})$ where this notation means view $\mathbb{C}$ as a $(\mathbb{C}, \mathbb{C})$-bimodule. □

So now $BPU(H)$ can be seen to be the moduli stack of algebras that look like $b(H)$, i.e. are isomorphic to $b(H)$ as algebras. These algebras are called 'type I factors,' i.e. infinite dimensional von Neumann algebras that are factors (the center is 1-dimensional) and the set of projections admits minimal elements. We've now eliminated the choice of H in the description of $K(\mathbb{Z}, 3)$.

You could replace $\mathbb{C}$ by $\mathbb{R}$ and nothing would change till the last paragraph, which would now have two isomorphism factors (one for matrices over $\mathbb{R}$ and one for matrices over the quaternions). So this would have a $\mathbb{Z}/2$ in π_0.

1.1. **Connections to twisted K-theory.** Suppose X is a space and there's a map from X to the stack of type I factors. This is equivalent to the data of a bundle over X whose fibers are of the type of $b(H)$. So you get a bundle of algebras over X.

K-theory is defined via bundles of vector spaces over X. The bundles above naturally define twisted K-theory.

2. A CHOICE-FREE $K(\mathbb{Z}, 4)$

Suppose E is a spectrum. Then E admits twistings by spherical fibrations. Given a spherical fibration, can apply $- \wedge E$ fiberwise and you get sections of that bundle of spectra. This is the twisted E-cohomology.

Now take $E = tmf$. Then there's a map $BO \to BGL_1 tmf$. You can precompose with $BString \to BO$ and the resulting map $BString \to BGL_1 tmf$ is null-homotopic. Thus, there's an extension to the cofiber $BO/BString \to BGL_1 tmf$. This cofiber has homotopy groups $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}$ in degrees 0,1,2,3,4. So there's a map $K(\mathbb{Z}, 4) \to BO/BString$.

In the background just now were type III factors (we will not say anything about type II factors). There's also a subtype called type III_1 factors. Let R be a hyperfinite type III_1 factor. Just like H, it turns out R is uniquely defined by this property, up to isomorphism. How do we construct R?

Consider $\mathbb{Q}^x \ltimes L^\infty(\mathbb{R})$ acting on $L^2(\mathbb{Q}^x \times \mathbb{R})$ via $((q, f) \cdot \zeta)(x, y) = f(y)\zeta(qx, qy)$. The closure is R.

Properties of R:

(1) $Z(R) = \mathbb{C}$, so R is a factor.

(2) $U(R) \simeq \{*\}$.

(3) $Aut(R) \simeq \{*\}$, but this is non-trivial.

We can now make a construction $0 \to ZU(R) \to U(R) \to PU(R) \to 0$ where $ZU(R) = S^1$.

Similarly, $0 \to Inn(R) \to Aut(R) \to Out(R) \to 0$.

Inner automorphisms are exactly the same as unitaries modulo those unitaries that act trivially (i.e. those from the center) so $Inn(R) = PU(R)$. Because $Aut(R)$ is contractible, this makes $Out(R)$ a $K(\mathbb{Z}, 3)$. Taking B of it yields a $K(\mathbb{Z}, 4) = BOut(R)$.

All the groups in the two exact sequences above are Polish groups, but $Out(R)$ is not a topological group because the map $Inn(R) \to Aut(R)$ is the inclusion of a dense subset.

Out(R) is still a sheaf of groups on Top. And $BOut(R)$ is $Bim(R)^x/iso$, i.e. invertible (R, R)-bimodules mod isomorphism.

Theorem 2.1. *Out(R) is the group of automorphisms of Bim(R), the monoidal category of (R, R)-bimodules. Again, these are automorphisms as a sheaf of spaces rather than as a group.*

The action of Aut(R) on Bim(R) is given by $\alpha \cdot M = ({}_\alpha M_\alpha)$, where α twists on both sides. We next need a trivialization of the action of Inn(R). That will induce a trivialization of the action of U(R) and we must be certain that this induces a trivialization of the action of ZU(R). So we take our formula for $\alpha \cdot M$ and we put $\alpha = ad(U)$ for $U \in U(R)$. So we need a trivialization $M \cong ({}_{ad(U)}M_{ad(U)})$. It turns out that the map $U \cdot (-) \cdot U^{-1}$ does the job. On ZU(R) this is the identity, as required, because $UU^{-1} = 1$.

We can now show a piece of the proof of the theorem, namely the part analogous to our proof of surjectivity in Theorem 1.1. This uses that Bim(R) is Morita equivalent to Bim($\mathbb{C}$). This Morita equivalence is given by ${}_{Bim(R)}R - Mod_{Bim(\mathbb{C}}$. Then you may formally copy the argument from Theorem 1.1. Given α in Aut(Bim(R)), use α to twist R-Mod (viewed as a (Bim(R),Bim($\mathbb{C}$))-bimodule) into R-Mod (viewed as a (Bim(R),α,Bim($\mathbb{C}$))-bimodule). The untwisting can be done in $\mathbb{C}$ and we get an equivalence M (equivariant with respect to the untwisting action) between the untwisted R-Mod and the twisted R-Mod. We then see that $\alpha = ad(M)$, proving surjectivity.

We finish by looking at BOut(R). This is the moduli stack of things that look like Bim(R). Hopefully this description will help in the project of finding geometric co-cycles for tmf. There should be a notion of bundle with an action of the bundle of categories.

①

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A $K(\mathbb{Z},4)$ in nature

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$K(\mathbb{Z},n)$'s

:n=0 • $\mathbb{Z}$

n=1 • S^1 or $\mathbb{C}^\times$

n=2 • $\mathbb{C}P^\infty$ $\quad \underbrace{\text{set of all}}_{\text{great}}$ lines in $\underbrace{\mathbb{C}^\infty}_{\text{artificial}}$

let's see if we like it…

B

$\rightsquigarrow$ set of all lines
stack

(could iterate B)

I won't go down that path today

n=3 • Let's think about other models of $K(\mathbb{Z},2)$ that I can use to get a model of $K(\mathbb{Z},3)$

Want a $K(\mathbb{Z},2)$ that's a group

need: Contractible group G with $S^1 \triangleleft G$.

$U(H) \simeq \{*\}$ (Kuiper) $\qquad$ H: Hilbert space

$PU(H) = U(H)/S^1$

$BPU(H) = K(\mathbb{Z},3)$

Note: Kuiper proves the contractibility of $U(H)$ in the norm topology. It's a bit of a difficult theorem, but it's the wrong theorem: the correct topology on $U(H)$ is the one that makes it a Polish group. Aside from the norm topology, the algebra $B(H)$ has a whole bunch of other topologies: they all agree on $U(H)$.

With respect to that topology, it's very easy to see that $U(H)$ is contractible

Given $u \in U(L[0,1])$ I'll construct a canonical path $1 \xrightarrow{u_t} u$

$$u_t : \underset{\substack{\circlearrowleft \\ u}}{L^2[0,t]} \oplus \underset{\substack{\circlearrowleft \\ 1}}{L^2[t,1]} \qquad \boxtimes$$

Let's see if we like it...

$U(H)$ depends on a choice of H (I'll adress that later)

$PU(H)$ ⇝ it's a quotient ☹ made to get the anwer we want

I want $PU(H) = \mathrm{Aut}(\cdots)$.

Thm: $PU(H) = \mathrm{Aut}(B(H))$ ← bounded operators on H

proof: • $PU(H)$ acts on $B(H)$

We need an action of $U(H)$ whose restriction to $S^1 \subset U(H)$ is trivial, and indeed the conjugation action has that property.

• Every automorphism $\alpha \in \mathrm{Aut}(B(H))$ is inner

KEEP

Use: $B(H) \simeq_M \mathbb{C}$

Not an algebraic Morita equivalence: it's analytical

~~$\alpha \rightsquigarrow {}_{B(H),\alpha}H_{\mathbb{C}}$~~

using the Morita equivalence, I can go back and forth between $B(H),\mathbb{C}$ - bimodules & $\mathbb{C}$-$\mathbb{C}$- bimodules.

Any two $\overset{\text{invertible}}{\vee}$ $\mathbb{C}$-$\mathbb{C}$-bimodules are isomorphic

$\Rightarrow$ the same holds for $B(H)$-$\mathbb{C}$-bimodules

$$\Rightarrow \boxed{ B(H)^{H}\mathbb{C} \xrightarrow[u]{\simeq} \overset{\alpha}{\overset{\rightsquigarrow}{B(H)_\alpha}}{}^{H}\mathbb{C} }$$

by writing out what it means for u to be a bimodule map $\Rightarrow$ $\alpha = \mathrm{ad}(u)$.

- $\mathrm{Ker}\left(U(H) \to \mathrm{Aut}(B(H))\right) \overset{?}{=} S^1$

$$\|$$

$Z(U(H))$

(identity bimodule)

$$Z(B(H)) = \mathrm{End}\left({}_{B(H)}B(H)_{B(H)}\right) \underset{\substack{\uparrow \\ B(H) \simeq_M \mathbb{C}}}{=} \mathrm{End}\left({}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\right) = \mathbb{C}$$

⊠

$BPU(H)$ = Moduli ~~space~~ stack of $B(H)$'s $\simeq K(\mathbb{Z},3)$

↑

"type I factors"

(can be axiomatized, and then it's a theorem that they are all isomorphic)

Note: I've eliminated the choice of H !

That construction is great: Can be modified to create other interesting homotopy types.

(what I have in mind:
- $\mathbb{C} \rightsquigarrow \mathbb{R}$
- add $\mathbb{Z}/2$-gradings)

Really the reason I like this model is because of the connection to twisted K-theory.
You might have heard that KU admits twists by $H^3(-,\mathbb{Z})$:

$$\begin{tikzcd} X \arrow[r] & {[\text{Stack of type I factors}]} \end{tikzcd}$$

$$\rightsquigarrow$$

$$\text{bundle of algebras over } X$$

If instead of looking at vector bundles over X you look at bundles of modules over this bundle of algebras, you get twisted K-theory.

Now really what I want to tell you about is a particular model of $K(\mathbb{Z},4)$ that bears a lot of analogies with the previous story.
The reason I'm very excited about having such a model of $K(\mathbb{Z},4)$ is because of the relation to twists of elliptic cohomology.

For any spectrum $\mathbb{E}$,
$\mathbb{E}$-cohomology admits twistings by vector bundles
(or, more generally, by spherical fibrations)

$$\begin{pmatrix}\text{spherical}\\ \text{fibration}\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{fiberwise}\\ -\wedge\mathbb{E}\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{sections of that bundle of spectra}\\ = \text{twisted } \mathbb{E} \text{ cohomology.}\end{pmatrix}$$

Now if $\mathbb{E}$ = tmf, by a theorem of Ando-Hopkins-Rezk, the map

$$\begin{tikzcd}
 & & K(\mathbb{Z},4) \arrow[d] \\
BString \arrow[r] \arrow[dr, Rightarrow] & BO \arrow[r] \arrow[d] & BO/BString \arrow[dl, dashed] \\
 & BGL_1(tmf) &
\end{tikzcd}$$

if I precompose it with

(which is to say)

(that's $\boxed{\mathbb{Z}\ \mathbb{Z}/2\ \mathbb{Z}/2\ 0\ \mathbb{Z}}$) 0 1 2 3 4

classifies twists of tmf

R: hyperfinite III_1 factor (v.N. algebra. unique up to isomorphism.)

Construction: $\mathbb{Q}^\times \ltimes L^\infty(\mathbb{R})$ acting on $L^2(\mathbb{Q}^\times \times \mathbb{R})$ by

(many, many constructions that look very different)

$$((q,f)\cdot\xi)(x,y) = f(y)\,\xi(qx,qy)$$

and take closure in w.o.t

- $Z(R) = \mathbb{C}$ "factor"
- $U(R) \simeq \{*\}$ (analog of Kuiper)
- $Aut(R) \simeq \{*\}$ [unpublished – Wassermann] (and not written up)

$$\begin{tikzcd}
0 \arrow[r] & \overset{\overset{S^1}{\parallel}}{U(Z(R))} \arrow[r] & \overset{\overset{\{*\}}{\wr\wr}}{U(R)} \arrow[r] & Inn(R) \arrow[r] & 0 \\
0 \arrow[r] & Inn(R) \arrow[r] & \overset{\overset{\{*\}}{\wr\wr}}{Aut(R)} \arrow[r] & Out(R) \arrow[r] & 0
\end{tikzcd}$$

All Polish groups

[dense image]

not a topological group.
It's a sheaf on Top
⇒ still has an associated homotopy type $\simeq K(\mathbb{Z},3)$

⑥

$$\Rightarrow B\,\mathrm{Out}(R) \simeq K(\mathbb{Z},4)$$

Next goal: Convince that that's a natural construction.

[Note: $\mathrm{Out}(R) = \mathrm{Bim}(R)^{\times}/_{\text{iso.}}$]

$\uparrow$ (invertible bimodules)

Claim: $\mathrm{Out}(R) = \mathrm{Aut}(\ldots)$

aut. as ⊗ category

$\mathrm{Bim}(R)$, the ⊗-category of RR-bimodules.

Sketch of proof First: $\mathrm{Out}(R)$ acts on $\mathrm{Bim}(R)$

An action of $\mathrm{Out}(R)$ on $\mathrm{Bim}(R)$ consists of:

- An action of $\mathrm{Aut}(R)$

  $\alpha \cdot M := {}_{\alpha}M_{\alpha}$ ← (Bimodule with both actions twisted by α)

- A trivialization of its restriction to $\mathrm{Inn}(R)$

→ A trivialization of the action of $\mathrm{Inn}(R)$ consists of:

- A trivialization of the action of $U(R)$

  $$M \xrightarrow[\simeq]{u\cdot(\ )\cdot u^{-1}} \underset{ad(u)\ \ ad(u)}{M}$$

- whose restriction to S^1 does nothing.

(1) Every automorphism of $\mathrm{Bim}(R)$ is inner [i.e. of the form $ad(M)$ for some $M \in \mathrm{Bim}(R)^{\times}$]

(2) $\mathrm{Out}(R) \longrightarrow \mathrm{Aut}(\mathrm{Bim}(R))$ injective

(3) $\mathrm{Aut}(\mathrm{Bim}(R))$ is a group (not a 2-group)
(it's zero-truncated)

I'll prove (1) for you:

Assumption: There exists an appropriate notion of Morita equivalence s.t. $\mathrm{Bim}(R) \simeq_M \mathrm{Bim}(\mathbb{C})$, and the equivalence bimodule is given by

$$ {}_{\mathrm{Bim}(R)}R\text{-Mod}_{\mathrm{Bim}(\mathbb{C})} $$

(Certainly not an algebraic notion of Morita equivalence).

Given $\alpha \in \mathrm{Aut}(\mathrm{Bim}(R))$, consider

$$ {}_{\mathrm{Bim}(R),\,\alpha}R\text{-Mod}_{\mathrm{Bim}(\mathbb{C})} $$

Using the Morita equivalence, I can go back and forth between Bim(R) - Bim(ℂ) - bimodules & Bim(ℂ) - Bim(ℂ) - bimodules. But the latter are rather boring things, and one can show that there is only one invertible Bim(ℂ) - Bim(ℂ) - bimodule up to isomorphism. ∴

$$ {}_{\mathrm{Bim}(R)}R\text{-Mod}_{\mathrm{Bim}(\mathbb{C})} \xrightarrow{\simeq} {}_{\mathrm{Bim}(R),\,\alpha}R\text{-Mod}_{\mathrm{Bim}(\mathbb{C})} $$

functor is given by some bimodule M

Writing out what it means for M to be a Bim(R) - Bim(ℂ) - bimodule map ⇒ $\alpha = \mathrm{ad}(M)$.