A VECTOR FIELD APPROACH FOR SHARP LOCAL WELL-POSEDNESS OF QUASILINEAR WAVE EQUATIONS

QIAN WANG

ABSTRACT. We give a proof of the sharp local well-posedness of general quasilinear wave equation by a vector field approach. The $H^{2+\epsilon}$ local well-posedness result was proved by Smith and Tataru by constructing parametrix using wave packet. Based on the commuting vector field approach, this type of results have been established by Klainerman and Rodnianski for $(3+1)$ Einstein vacuum equations, by taking advantage of $\mathbf{Ric} = 0$. However, $\mathbf{Ric}$ of the metric of the general type of equations contains second order derivatives of the metric, which makes it very difficult to implement the vector field approach to derive the sharp result. To get around this difficulty, our idea is to employ a conformal method to modify the spacetime metric locally such that the second order term in $\mathbf{Ric}(L, L)$ vanishes, with L the tangential vector field of the null pair $\{L, \underline{L}\}$. The null hypersurface of the modified metric is then smooth enough to provide a bounded Morawetz type energy. The other difficulty we encounter is that we only have very limited control on derivatives of the conformal factor. This leads to worse behavior of the deformation tensor ${}^{(K)}\pi$ of Morawetz vector field K under the conformal metric. The part of ${}^{(K)}\pi$ coming from $\underline{L}$ in the Morawetz vector field becomes worse while the part coming from L behaves better via the conformal change of the metric. If using the standard approach to derive Morawetz energy, these two parts will inevitably appear simultaneously. We solve this issue by adapting the new physical approach by Dafermos and Rodnianski to derive Morawetz energy by using vector fields involving L only.