A VECTOR FIELD APPROACH FOR SHARP LOCAL WELL-POSEDNESS OF QUASILINEAR WAVE EQUATIONS

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ϕ may be rough. (pg 1)

The over all goal of her talk/work is to take the methods used by others to prove sharp local well-posedness of the Einstein evolution equations ($s > 2$ for the initial data in $H^s \times H^{s-1}$), and apply them to prove such sharp local well-posedness for general hyperbolic quasilinear equations. She then goes into many details of the method of proof.

On board about 30 min in: Start with non-linear equation:

$$\Box_{g(\phi)} phi = N(\phi, \partial\phi).$$

A vector field approach for sharp Local well-posedness of quasilinear wave equations

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Consider the general quasilinear wave equation,

$$\begin{cases} \Box_{\mathbf{g}(\phi)}\phi := \partial_t^2\phi - g^{ij}(\phi)\partial_i\partial_j\phi = N(\phi, \partial\phi), \\ \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1 \end{cases} \tag{1}$$

where $N(\phi, \partial\phi) = \sum_{\alpha,\beta} N^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi$, g is uniformly elliptic, g and $N^{\alpha\beta}$ are smooth with respect to their arguments.
Here $\mathbf{g}$ the spacetime metric is $-dt^2 + g_{ij}(\phi)dx^i dx^j$.

An important example: Einstein vacuum equations

$$\mathbf{Ric}(\mathbf{g}) = 0 \tag{EV}$$

under wave coordinates $\{x^\alpha\}$: $\Box_{\mathbf{g}} x^\alpha = 0, \alpha = 0, \cdots, 3$ take the form

$$\Box_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g}), \alpha, \beta = 0, 1, 2, 3.$$

where $\mathcal{N}_{\alpha\beta}$ is quadratic in $\partial\mathbf{g}$.

The local well-posedness with respect to the regularity of $(\phi_0, \phi_1) \in H^s \times H^{s-1}$ has been considered by many authors:

- $s > 4$: Choquet-Bruhat, Acta Math. 1952
- $s > \frac{5}{2}$: Hughes-Kato-Marsden Arch. Rat. Mech. Anal., 1977
- $s > 2 + \frac{1}{4}$: Bahouri-Chemin, Amer J Math 1999 and Tataru, Amer J Math, 2000.
- $s > 2 + \frac{1}{6}$: Tataru, J Amer Math Soc, 2002.
- $s > 2 + \frac{2-\sqrt{3}}{2}$: Klainerman-Rodnianski, Duke Math J, 2003. (Commuting vector field approach)

  $s > 2$: for Einstein vacuum equations, Klainerman-Rodnianski, Ann of Math, 2005

- $s > 2$: Tataru and Smith, Ann of Math, 2005. (Constructing parametrix using wave packets). This is a sharp result due to the counter example by Lindblad.
- $s > 2$: Wang, arXiv:1201.0049, Einstein vacuum equations with CMCSH gauge (vector fields approach)
- L^2 conjecture: For Einstein vacuum equations the local well-posedness holds for $s = 2$ in Coulomb gauge with maximal foliation. (Klainerman, Rodnianski and Szeftel, Szeftel, arXiv:1204.1767-1204.1772$\cdots$)
- Question: Is it possible to achieve $s > 2$ result for general quasilinear wave (1) by vector fields approach?

- The results with $2 < s \leq \frac{5}{2}$ are all established by Strichartz estimate. Why do we need to do Strichartz estimate?
- How to use commuting vector fields approach for wave equations to establish Strichartz estimate?
- Why is **Ric** involved?

Theorem 1

Consider quasilinear initial value problem (1) in $\mathbb{R}^{3+1}$. For any $s > 2$ and $M_0 > 0$, there exist positive constants T and M_1 such that, with $I := [-T, T]$, there hold

(i) *For any initial data set (ϕ_0, ϕ_1) with $\|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} \le M_0$, there exists a unique solution $(\phi_0, \phi_1) \in C(I, H^s \times H^{s-1}) \times C^1(I, H^{s-1} \times H^{s-2})$ to (1)*

(ii) *There holds*

$$\|\partial\phi\|_{L_I^2 L_x^\infty} + \|\partial\phi\|_{L_I^\infty H^{s-1}} \le M_1.$$

Main steps

- Bootstrap assumptions

$$\int_{-T}^{T} \|\partial\phi\|_{L_x^\infty} dt \le B_1.$$

- Energy estimates $\|\partial\phi\|_{L_t^\infty H^{s-1}} \le C$.
- **Strichartz estimates**$\Rightarrow$ Improvement of Bootstrap assumptions: $\exists\delta > 0$, there holds

$$\|\partial\phi\|_{L^2_{[0,T]} L_x^\infty} \lesssim T^\delta.$$

- Consider $\|P_\lambda \partial\phi\|_{L^2_I L^\infty_x}$ where $\sum_\lambda P_\lambda = Id$. Prove

$$\|P_\lambda \partial\phi\|_{L^2_I L^\infty_x} \lesssim \lambda^{-\delta_*} T^{\frac{1}{2}-\frac{1}{q}} \|\partial\phi\|_{H^{s-1}(\Sigma_0)} \tag{2}$$

where $q > 2$ is very close to 2. Then sum over λ.

Remarks:

- P_λ is the Littlewood-Paley projector with frequency $\lambda = 2^k$ defined for any function f by

$$P_\lambda f(x) = f_\lambda(x) = \int e^{ix\cdot\xi}\Psi(\lambda^{-1}\xi)\hat{f}(\xi)d\xi$$

and Ψ is a smooth bump function supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ and $\sum_{k\in\mathbb{Z}} \Psi(2^k\xi) = 1$, for $\xi \neq 0$. We denote $f_{\leq\lambda} := \sum_{\mu\leq\lambda} P_\mu f$.

Finite band:

$$\partial P_\lambda f \sim \lambda P_\lambda f,$$

Bernstein inequality:

$$\|P_\lambda f\|_{L_x^p} \lesssim \lambda^{\frac{3}{q}-\frac{3}{p}} \|P_\lambda f\|_{L_x^q}, q < p \in [1, \infty].$$

- An attempt to consider (2): Using Bernstein inequality

$$\|P_\lambda \partial\phi\|_{L_I^2 L_x^\infty} \lesssim |I|^{\frac{1}{2}} \lambda^{\frac{3}{2}} \|P_\lambda \partial\phi(t)\|_{L_x^2}.$$

  Sum over λ,

$$\sum_\lambda \|P_\lambda \partial\phi\|_{L_I^2 L_x^\infty} \lesssim |I|^{\frac{1}{2}} \|\partial\phi(t)\|_{??}$$

  with the norm either $B^{\frac{3}{2}}_{2,1} or\ H^{\frac{3}{2}+}$, at the level of $H^{\frac{5}{2}+}$ in terms of metric.
  Then use the energy estimate to pass from Σ_t to the initial slice Σ_0.
  It requires the data $(\phi, \phi_1) \in H^{\frac{5}{2}+} \times H^{\frac{3}{2}+}$.
- (2) surpasses the above approach by $\frac{1}{2}$ derivative.

Strichartz estimates for wave equations

Consider

$$\Box\phi = 0 \text{ on } \mathbb{R}^{1+n}, \ \phi[0] = (\phi_0, \phi_1) \tag{3}$$

there holds

- $$\|\phi\|_{L_t^q L_x^r} \lesssim \|\phi[0]\|_{\dot{H}^s}$$

  where $\frac{n}{2} - s = \frac{1}{q} + \frac{n}{r}$ and (q, r) is wave admissible, i.e.

  $$2 \le q \le \infty, \, 2 \le r < \infty, \text{ and } \frac{2}{q} \le \frac{n-1}{2}(1 - \frac{2}{r})$$

- $$\|\phi\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^3)} \lesssim \|\phi[0]\|_{H^{1+\epsilon}}$$

Standard approach

- $\hat{\phi}(t,\xi)$ is a linear combination of $e^{\pm it|\xi|}\widehat{f}(\xi)$, $e^{\pm it|\xi|}\widehat{g}(\xi)/|\xi|$.
- Prove for one Littlewood-Paley piece,

$$\|P_1\phi\|_{L_t^q L_x^r} \leq C\|P_1\phi[0]\|_{L_x^2},$$

with C independent of frequency $\lambda = 2^k$.
Rescale coordinates by λ and sum over the estimates for $P_\lambda\phi$.

- $\mathcal{T}\mathcal{T}^\star$ argument:
  (1) Define $\mathcal{T} : L_x^2 \to L_t^q L_x^r$

$$\mathcal{T}f(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi + it|\xi|}\beta(\xi)\hat{f}(\xi)d\xi = P_1\phi(t,0,(f,0))$$

where β is smooth radial function, $\mathrm{supp}\beta \subset \{\frac{1}{2} < |\zeta| < 2\}$.

  (2) $\|\mathcal{T}\mathcal{T}^\star\|_{L_t^{q'} L_x^{r'} \Rightarrow L_t^q L_x^r} = M^2 \Rightarrow \|\mathcal{T}F\|_{L_t^q L_x^r} \leq M\|F\|_{L_x^2}$ with M a constant independent of frequency.

(3) $\mathcal{T}\mathcal{T}^\star F = K * F$ where $K(t,x) = \int_{\mathbb{R}^n} e^{i(t|\xi|+x\cdot\xi)}|\beta|^2 d\xi$.

- Dispersive estimate: Consider $\|K * F\|_{L_t^q L_x^r}$ by interpolating between L^2 estimate and

$$\|K(t-s,\cdot) * F(s,\cdot)\|_{L_x^\infty} \lesssim (1+|t-s|)^{-\frac{n-1}{2}} \|F(s,\cdot)\|_{L_x^1}$$

Integrate in t with the help of Hardy-Littlewood-Sobolev.

Key ingredients : Fourier representation of solution, $\mathcal{T}\mathcal{T}^\star$ and dispersive estimate.

• For wave equations with varied coefficients, finding a Fourier representation of solution of $\Box_{\mathbf{g}}\phi = 0$ causes a big problem.

$\mathcal{T}\mathcal{T}^*$ argument without Fourier representation

- $\mathcal{H} := \{I = (i_0, i_1) \text{ with } i_0, i_1 \in L^2(\mathbb{R}^n)\}$. The scalar product of $\mathcal{H}$ is defined by

$$\langle I, J\rangle = \int_{\Sigma_0} (i_1 \cdot j_1 + \delta^{ab}\nabla_a i_0 \cdot \nabla_a j_0) dx$$

- $\mathcal{T}(I) := -P\partial_t W(t, 0, I[0])$ where $I[s] = (f, h)$ and $W(t, s, I[s])$ is the solution of

$$\Box\phi = 0, \phi(s, s) = f, \partial_t\phi(s, s) = h.$$

Then

$$\mathcal{T}\mathcal{T}^\star F = \int_0^{t_*} P\partial_t W(t, s, (0, -PF)) ds + \text{remainder}.$$

- Works for $\Box_{\mathbf{g}}\phi = 0$. But we need to control remainder.

- The remainder involves deformation tensor ${}^{(\mathbf{T})}\pi = \mathcal{L}_{\mathbf{T}}\mathbf{g}$, whose components are $k = -\frac{1}{2}\partial_t g$.
- Strategy: Bootstrap argument and $M^2 \leq C + \frac{1}{2}M^2$, with the remainder incorporated as the last term, which implies $M^2 \leq 2C$.
- For fixed large frequency λ, we partition time interval such that $\|{}^{(\mathbf{T})}\pi\|_{L_t^1 L_x^\infty}$ is appropriately controlled in each subintervals I_*.

- * Consider homogeneous equation $\Box_{\mathbf{g}}\phi = 0$. The inhomogeneity is treated by Duhamel's principle.
- Dyadic Strichartz: Employ $\mathcal{T}\mathcal{T}^\star$ argument to prove that there holds for solution of $\Box_{\mathbf{g}}\phi = 0$,

$$\|P_\lambda \partial\phi\|_{L^q_{I_*} L^\infty_x} \lesssim \lambda^{\frac{1}{2}-\frac{1}{q}+1}\|\phi[0]\|_{H^1}$$

where $q > 2$. We then apply it to frequency "localized" data.

- Decay Estimate $\Rightarrow$ Dyadic Strichartz. Rescale $(t, x) \to (\frac{t}{\lambda}, \frac{x}{\lambda})$. Prove the decay estimate

$$\|P\partial_t\phi(t)\|_{L^\infty_x} \le \left(\frac{1}{(1+t)^{\frac{2}{q}}} + d(t)\right)(\|\phi[0]\|_{H^1} + \|\phi(0)\|_{L^2})$$

with q and $d(t)$ good enough for using Hardy-Littlewood-Sobolev.

- **Key part** Control Morawetz energy. (vector fields approach)
  Morawetz energy $\Rightarrow$ Decay estimate.
- Sum over all time subintervals I_*.

Commuting vector field approach for dispersive estimate

(Morawetz, John, Klainerman, Christodoulou, etc)

- The energy-momentum tensor associated to a solution ϕ of $\Box\phi = 0$ is
$$Q_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\mathbf{m}_{\alpha\beta}\mathbf{m}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$
Consider the generalized energy
$$\mathcal{C}[\phi] = \int_{\mathbb{R}^3} Q_{\alpha\beta}X^\alpha{\partial_t}^\beta + \text{ modification term for treating } f$$
where X is a timelike conformal killing vector field.
- Define ${}^{(X)}\pi_{\mu\nu} := \mathbf{D}_\mu X_\nu + \mathbf{D}_\nu X_\mu$. A vector fields $X = X^\mu\partial_\mu$ in $(\mathbb{R}^{1+n}, \mathbf{m})$ is conformal Killing if there is a function f such that ${}^{(X)}\pi_{\mu\nu} = f\mathbf{m}_{\mu\nu}$.
If $f = 0$, then X is a Killing vector fields.
- Use Morawetz vector field: $\frac{1}{2}\left((t-r)^2(\partial_t - \partial_r) + (t+r)^2(\partial_t + \partial_r)\right)$. Then derive dispersive estimates by controlling $\mathcal{C}[\phi]$ via Sobolev type estimates.

- In the work of the global stability of Minkowski spacetime (Christodoulou-Klainerman, 93), all conformal Killing vector fields in $(\mathbb{R}^{1+3}, \mathbf{m})$ are generalized in terms of null frames. Due to small data, they are approximately (conformal) Killing, i.e. ${}^{(X)}\pi - f\mathbf{g}$ small in appropriate norms.
- In large data problem, we do not expect these quantities to be small. Working in the rescaled coordinates $(t, x) \to (\frac{t}{\lambda}, \frac{x}{\lambda})$, the spacetime is stretched like Minkowski spacetime. Then we give estimates on ${}^{(X)}\pi - f\mathbf{g}$ in terms of frequency.
- How many orders of the conformal energy do we need? When the background is rough, it can be very challenging to control even just the lowest order energy.

Treatments on the background metric

- Bahouri-Chemin, Tataru, Klainerman-Rodnianski ($a < 1$)

$$P_{\lambda^a}(\Box_{\mathbf{g}(\phi)}\phi) = P_{\lambda^a}(N(\phi, \partial\phi)) \Rightarrow \Box_{\mathbf{g}_{\leq\lambda^a}} P_\lambda \phi = \text{Remainder}.$$

where $g_{\leq\lambda^a} = P_{\leq\lambda^a} g^{ij}(P_{\leq\lambda^a}\phi)$ and $\mathbf{g}_{\leq\lambda^a} = (-1, g_{\leq\lambda^a})$.
For $s > 2$ result, consider Strichartz estimate for with $a = 1$
$\Box_{\mathbf{g}_{\leq\lambda}}\phi = 0$.

- Derivatives of $h := \mathbf{g}_{\leq\lambda}$ can be derived, which take the form of λ^b.
- **DRic** becomes the crucial difficulty for controlling ${}^{(X)}\pi$. If treating Einstein vacuum equation, this procedure causes $\mathbf{R}_{\alpha\beta}(\mathbf{g}_{\leq\lambda}) \neq 0$. To control the defected Ricci is a delicate procedure and could cause a big hurdle.

For Einstein equation, without smoothing $\mathbf{g}$, we manage to establish Strichartz estimates for $\Box_{\mathbf{g}}\phi = 0$. (Wang, 2012)

- Under Einstein vacuum metric $\mathbf{g}$, **Ric** vanish. But differentiability of ${}^{(X)}\pi$ are limited.
- Strategy:
  - Prove the lowest order conformal energy $\Rightarrow$ the desired dispersive estimates. Then we need less control ${}^{(X)}\pi$.
  - To control ${}^{(X)}\pi$, we squeeze out a bit more differentiability from Strichartz estimates to beat the log-loss in Calderon-Zygmund theory.

Main steps for dispersive estimates

Spatial Localization

In rescaled coordinate $(t,x) \to (\frac{t}{\lambda}, \frac{x}{\lambda})$, we consider $\Box_h \phi = 0$. let $\{\chi_J\}$ be the partition of unity on Σ_{t_0} subordinating to the cover $\{B_J\}$, essentially disjoint, and $\text{supp}\chi_J \subset B_{3/4}$.

$$\Box_h \phi_J = 0, \phi_J(t_0) = \chi_J \cdot \phi(t_0), \partial_t \phi_J(t_0) = \chi_J \cdot \partial_t \phi(t_0), \ t_0 \approx 1.$$

Then $\phi(t) = \sum_J \phi_J(t)$. It suffices to consider decay estimate for $\phi_J(t)$ followed with combining all pieces together.

Essentially disjoint: Any ball in $\{B_J\}$ intersect at most 10 other balls.

Σ_{t_0} is the level set of $t = t_0$.

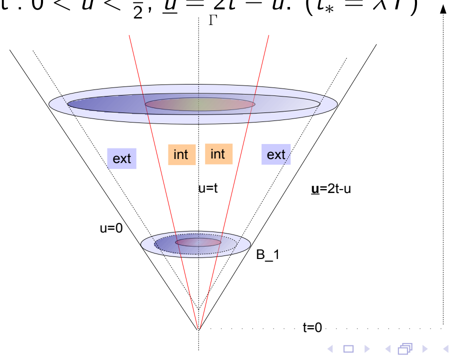
Conformal energy Consider, for $t \in [1, t_*]$, the conformal type energy for ϕ whose support is within $\mathcal{D}^+(B_1)$.

$$\mathcal{C}[\phi](t) = \int_{\mathbb{R}^3} \underline{u}^2(|\nabla\!\!\!/ \phi|^2 + |L(\phi)|^2) + u^2|\underline{L}(\phi)|^2 + |\phi|^2\} d\mu$$

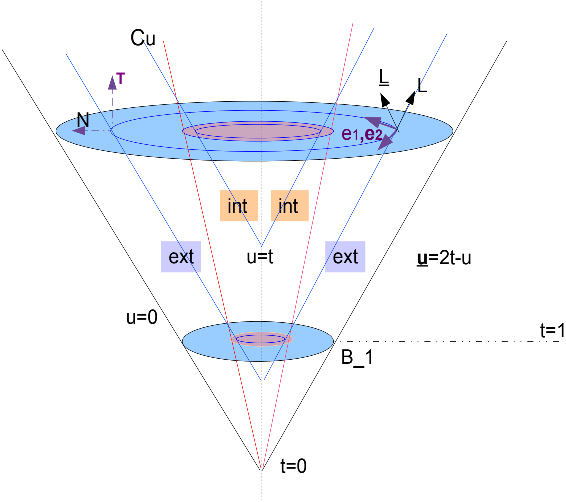
where u, the optical function, is defined by

$$h^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \;\; u = t \;\text{ on time axis } \Gamma.$$

$\text{Int}: \frac{t}{2} < u < t$; $\text{Ext}: 0 < u < \frac{t}{2}$; $\underline{u} = 2t - u$. $(t_* = \lambda T)$



- Foliate $\mathcal{D}^+(B_1)$ by null cones C_u. $S_{t,u} := C_u \cap \Sigma_t$.
- "Radial foliation": $\mathcal{D}^+(B_1) \cap \Sigma_t = \cup_{0 \le u \le t} S_{t,u}$.
- Null frame: $\{L, \underline{L}, e_1, e_2\}$, where $\{e_1, e_2\}$ is the orthonormal frame on $S_{t,u}$, and $L = \partial_t + N$, $\underline{L} = \partial_t - N$.



Bounded conformal energy

Prove Boundedness theorem: $\forall 1 \le t \le t_*$, $t_* = \lambda T$, for the solution of $\Box_h \phi = 0$, there holds

$$\mathcal{C}[\phi](t) \lesssim \mathcal{C}[\phi](1).$$

Morawetz vector field: $X = \frac{1}{2}(u^2 \underline{L} + \underline{u}^2 L)$.

Let $P_\alpha = Q_{\alpha\beta}[\phi] X^\beta$ where $Q_{\alpha\beta}$ is the energy momentum tensor of $\Box_h$. Using

$$\mathbf{D}^\alpha P_\alpha = \frac{1}{2} Q^{\alpha\beta(X)} \pi_{\alpha\beta} + X\phi \cdot \Box_h \phi,$$

with suitable normalization ($P_\alpha \to \bar{P}_\alpha$), we have $1 < t < t_*$

$$\int_{\Sigma_t} \bar{P}_\alpha (\partial_t)^\alpha - \int_{\Sigma_1} \bar{P}_\alpha (\partial_t)^\alpha = \int_{\Sigma \times I} \mathbf{D}^\alpha \bar{P}_\alpha$$
$$= \int \frac{1}{2} Q^{\alpha\beta(X)} \bar{\pi}_{\alpha\beta} + \Box_h \phi \cdot X\phi + l.o.t$$

- To control deformation tensor ${}^{(X)}\bar{\pi}$ in $\mathcal{D}^+(B_1)$, we employ null cone foliations and study connection coefficients of null frame. This part relies on **Ric** and smoothness of metric crucially.
-

$$[\Box_h, Y]\phi = \mathbf{D}_\alpha{}^{(Y)}\pi^{\alpha\beta}\mathbf{D}^\beta\phi - \frac{1}{2}\mathbf{D}^\beta \mathrm{tr}^{(Y)}\pi \cdot \mathbf{D}_\beta\phi + {}^{(Y)}\pi^{\alpha\beta}\mathbf{D}^2_{\alpha\beta}\phi,$$

Higher order conformal energy $\mathcal{C}[Y\phi]$ requires more differentiability of ${}^{(Y)}\pi^{\alpha\beta}$,

Conformal energy ⇒ Decay estimate

• We obtain the decay estimate with merely the control of $\mathcal{C}[\phi]$, by taking advantage of $P\partial_t\phi$ has frequency ≈ 1.

- Goal: Modulo certain loss,

$$(t+1)\|P\partial_t\phi(t,x)\|_{L^\infty_x} \lesssim \mathcal{C}[\phi](t) + \cdots .$$

- $P\partial_t\phi(t,x)$ has no spatial compact support ($\because$ uncertainty principle). We can not establish radial foliation globally.
- Localizing it by dropping "P" requires high order conformal energies.
- We localize $P\partial_t\phi$ within a region, around $\mathcal{D}^+(B_1)$ and having valid foliation. Then control $P\partial_t\phi$ in terms of the $\|\underline{u}(\nabla\!\!\!/\phi, L(\phi))\|_{L^2(\Sigma)}$ term in $\mathcal{C}[\phi](t)$.

$$
\begin{aligned}
P\partial_t\phi &= P\left(\varpi\cdot((L-N)\psi)\right) + \text{“interior” part} \\
&= P\varpi L\phi + \varpi N^j P\partial_j(\phi) + [P,\varpi N^j]\partial_j\phi + \cdots \\
&= I + II + III + \cdots
\end{aligned}
$$

where ϖ a cut-off function, essentially supported in the exterior part of domain of influence.

- Term III: Commutator estimates require control of $\partial_j N$, which is reduced to control of connection coefficient, where $\nabla_N N$ is a difficult quantity. For (1), we do not have good control on it.
- For (1), commuting has to be avoided. We employ duality argument and integration by part to solve the issue.
-

$$
I: \quad \|P\varpi L\phi\|_{L^\infty_x} \lesssim \|L\phi\|_{L^2_x}, \tag{4}
$$

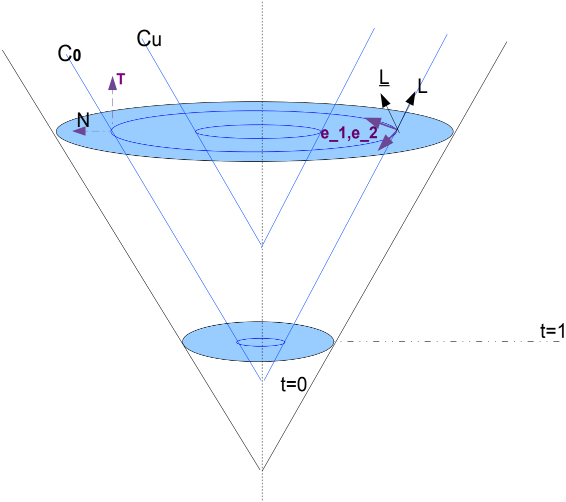
$$
II: \|\varpi\tilde{P}\phi\|_{L^\infty_u L^\infty(S_{t,u})} \lesssim r^{\frac{\delta}{2}}(\|P\varpi\nabla\!\!\!/\phi\|_{H^1} + \|[P,\varpi\nabla\!\!\!/]\phi\|_{H^1} + \cdots) \tag{5}
$$

where we employed Sobolev embedding and trace inequality. For (1), same issue occurs for II.

Foliations by Weakly regular null hypersurfaces

• $\cup_u C_u$. To control ${}^{(X)}\pi$, we need to control the following connection coefficients,
With null frame $\{L = e_4, \underline{L} = e_3, e_1, e_2\}$ define

$$\chi_{AB} = \langle \mathbf{D}_A e_4, e_B \rangle, \qquad \underline{\chi}_{AB} = \langle \mathbf{D}_A e_3, e_B \rangle$$
$$\zeta_A = \frac{1}{2}\langle \mathbf{D}_3 e_4, e_A \rangle, \qquad \underline{\zeta}_A = \frac{1}{2}\langle \mathbf{D}_4 e_3, e_A \rangle$$
$$\xi_A = \frac{1}{2}\langle \mathbf{D}_3 e_3, e_A \rangle.$$


Co
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e_1,e_2
t=1
t=0

Null structure equations

$$L\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2 - \bar{k}_{NN}\mathrm{tr}\chi - \mathbf{R}_{44} \quad (6)$$

$$\frac{d}{ds}\nabla\!\!\!/\,\mathrm{tr}\chi + \frac{3}{2}\mathrm{tr}\chi\nabla\!\!\!/\,\mathrm{tr}\chi = \nabla\!\!\!/\,\mathbf{R}_{44} - \hat{\chi}\cdot\nabla\!\!\!/\,\mathrm{tr}\chi - 2\hat{\chi}\cdot\nabla\!\!\!/\,\hat{\chi} - (\zeta + \underline{\zeta})|\hat{\chi}|^2 + \cdots \quad (7)$$

$$(\mathrm{div}\,\hat{\chi})_A + \hat{\chi}_{AB}\cdot k_{BN} = \frac{1}{2}(\nabla\!\!\!/\,\mathrm{tr}\chi + k_{AN}\mathrm{tr}\chi) - \mathbf{R}_{B4AB} \quad (8)$$

$$\mathrm{div}\,\zeta = \frac{1}{2}(\mu + 2N\log n\mathrm{tr}\chi - 2|\zeta|^2 - |\hat{\chi}|^2 - 2k_{AB}\chi_{AB}) - \delta^{AB}\mathbf{R}_{A43B} \quad (9)$$

$$\mathrm{curl}\,\zeta = \frac{1}{2}\epsilon^{AB}k_{AC}\hat{\chi}_{CB} - \frac{1}{2}\epsilon^{AB}\mathbf{R}_{B43A} \quad (10)$$

$$L\mu + \mathrm{tr}\chi\mu = -\underline{L}\mathbf{R}_{44} - \mathrm{tr}\chi\mathbf{R}_{34} + 2\hat{\chi}\cdot\nabla\!\!\!/\,\zeta + (\zeta - \underline{\zeta})\cdot(\nabla\!\!\!/\,\mathrm{tr}\chi + \mathrm{tr}\chi\zeta) \quad (11)$$
$$- \frac{1}{2}\mathrm{tr}\chi(\hat{\chi}\cdot\underline{\hat{\chi}} - 2\rho + 2\underline{\zeta}\cdot\zeta) + \cdots$$

where $\mu = \underline{L}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}$.

Strategy:

- Goal: to show $\hat{\chi}, \zeta$ have the strichartz type estimates as ∂h.
- Characteristic energy estimates: For $\Box\phi = 0$ in $(\mathbb{R}^{3+1}, m)$, for null cone initiating from $t = t_0$ ending at t_1

$$\text{“Flux”}: \quad \|\nabla\!\!\!/\phi, L\phi\|_{L^2(C_u)} \lesssim (\int_{t=t_1} |\mathbf{D}\phi|^2 dx)^{\frac{1}{2}}$$

  In the same spirit, we will use $\Box_h \partial h = \cdots$ to give bound on $\nabla\!\!\!/(\partial h), L(\partial h)$ along C_u with energy estimate.
- Flux gives the control over $\nabla\!\!\!/\mathrm{tr}\chi$ and thus the induced metric of $S_{t,u}$. Then we can justify L^p type Calderon-Zygmund theorem.
- Use Hodge system

$$\mathrm{div}\,\hat{\chi} = \nabla\!\!\!/\partial h + \nabla\!\!\!/\mathrm{tr}\chi + \cdots$$

  and Calderon Zygmund theory to control “$\hat{\chi} - \partial h$” in terms of flux.
- The proof relies on bootstrap arguments.

$s > 2$ for General quasilinear wave (1)

• Among the null structure equations, the linear terms in **DRic** are the worst terms.

- To bound $\mathcal{C}[\phi]$, we hope to

$$\|\nabla\!\!\!/\, \mathrm{tr}\chi\|_{L_t^\infty L_S^2} \lesssim \|\nabla\!\!\!/\, \partial h, L\partial h\|_{L^2(C_u)}.$$

  But what we obtain from (7) is

$$\|r^{\frac{1}{2}} \nabla\!\!\!/\, \mathrm{tr}\chi\|_{L_t^\infty L_S^2} \lesssim \|r \nabla\!\!\!/\, \mathbf{R}_{44}\|_{L^2(C_u)}$$

  where the extra r causes could be a huge number due to rescaling. In this step, the loss is equivalent to one derivative.

- $\nabla\!\!\!/\, \mathbf{R}_{44} = \nabla\!\!\!/\, \partial^2 h + \cdots$, it has the same estimate as $\nabla\!\!\!/\, \partial h$ in $L^2(C_u)$, which benefits from smoothing $\mathbf{g}$ to h followed with rescaling.
- We have similar issue with μ. And $\underline{L}\mathbf{R}_{44}$ is worse than $\nabla\!\!\!/\, \mathbf{R}_{44}$.

- Klainerman-Rodnianski: For Einstein vacuum equation, $\mathbf{Ric}(h)$, $\nabla\mathbf{Ric}(h) \approx 0$. Nevertheless, in general, we do not expect $\mathbf{Ric}$ to be good.
- We have the following decomposition

$$\mathbf{R}_{\alpha\beta} = -\frac{1}{2}\Box_h h_{\alpha\beta} + \frac{1}{2}(\mathbf{D}_\alpha V_\beta + \mathbf{D}_\beta V_\alpha) + S_{\alpha\beta}$$

  with a one form $V \approx h \cdot \Gamma$, and a symmetric two tensor $S \approx h \cdot \Gamma \cdot \Gamma$ Using wave equation for h, $\mathbf{R}_{\alpha\beta} = \frac{1}{2}(\mathbf{D}_\alpha V_\beta + \mathbf{D}_\beta V_\alpha) + h \cdot \Gamma \cdot \Gamma$
- Main difficulties are tied to treating derivative of $\mathbf{R}_{44}$. We have to "dissolve" the $\mathbf{D}V$ part in $\mathbf{R}_{44}$, which is the second derivative of the metric h.
- We employ conformal method. Optical function and null cones are invariant under conformal change of metric. $\widetilde{\mathrm{tr}\chi} \doteq \mathrm{tr}\chi + V_4$.

Consider conformal energy of solution of $\Box_h \phi = 0$ in the domain $\mathcal{D}^+$.

Let $h_{\mu\nu} = \Omega^2 \tilde{h}_{\mu\nu}$ with the conformal factor $\Omega = e^{-\sigma}$. There holds with $\tilde{\phi} = \Omega\phi$, $\Box_h \phi - \frac{1}{6}\mathbf{R}\phi = \Omega^{-3}(\Box_{\tilde{h}}\tilde{\phi} - \frac{1}{6}\tilde{\mathbf{R}}\tilde{\phi})$ hence

$$\Box_h \phi - \Omega^{-3}\Box_{\tilde{h}}\tilde{\phi} = (\Box_h \sigma + \mathbf{D}^\mu \sigma \mathbf{D}_\mu \sigma)\phi.$$

We control conformal energy of $\tilde{\phi}$ by using the equation of $\Box_{\tilde{h}}\tilde{\phi} = \cdots$

- Under the new metric, $\mathrm{tr}\widetilde{\chi} = \mathrm{tr}\chi + 2L\sigma$. Thus we set $L\sigma = \frac{1}{2}V_4$.
- The terms of ${}^{(\underline{u}^2 L)}\tilde{\pi}$ become better.
  (1) We can control $\nabla\!\!\!/ \mathrm{tr}\widetilde{\chi}, \hat{\chi}$ as desired.
  (2) The $\tilde{\zeta}$ under the conformal metric is connected to $\mu + 2\Delta\!\!\!/\sigma$

$$\mathrm{div}\,\tilde{\zeta} = \frac{1}{2}(\mu + 2\Delta\!\!\!/\sigma) + \delta^{AB}\mathbf{R}_{A43B} + \cdots \tag{12}$$

and $L(\mu + 2\Delta\!\!\!/\sigma) + \mathrm{tr}\chi(\mu + 2\Delta\!\!\!/\sigma) = \Box_h V_4 + \cdots$. This gives good estimate for $\tilde{\zeta}$, which is needed to control conformal energy.

- However the terms of ${}^{(u^2 \underline{L})}\tilde{\pi}$ get worse because $\underline{L}\sigma$ is involved. This drives us to we employ an approach to bound conformal energy, without using Morawetz vector fields K.
- Adapt the new physical approach by Dafermos and Rodnianski to get the Morawetz energy by using the vector field $r^p L$ only.
  (1) To obtain the integral type energy estimate, the standard procedure devised by Morawetz did rely on N in the domain of influence, we manage to use L.
  (2) We use the vector field N in a very small cylinder region near the time axis, which does not require much regularity of the background metric.