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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Dilts **Email/Phone:** JDilts@uoregon.edu

Speaker's Name: Pieter Blue

Talk Title: Introduction to decay of fields outside black holes

Date: 9 / 10 / 13 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Black hole stability, model equation, decay of vector fields

Please summarize the lecture in 5 or fewer sentances: He talked of the explicit black hole solutions, then talked of the proof of Minkowski stability. He then talked about how that method does and does not apply in proving stability for Kerr. He began providing details.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Blue - decay of fields

Fig 1

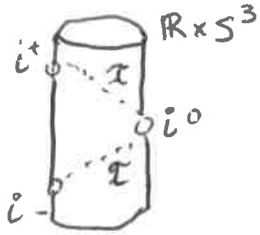


Fig 2

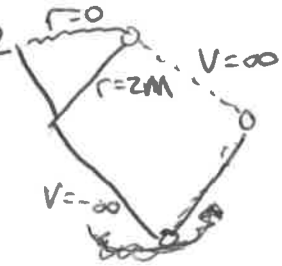


Fig 3

Fig 4

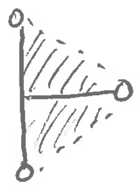
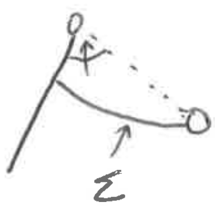


Fig 5



INTRODUCTION TO DECAY OF FIELDS OUTSIDE BLACK HOLES

PIETER BLUE

1. Explicit Black Hole Solutions

Let (M, g) be a spacetime. We say it is asymptotically flat (AF) if there exists an open $U \subset M$, and $R > 0$, and $\phi : \{(t, x, y, z) : |\vec{x}| > R\} \to U$ such that $\phi_* g_{\alpha\beta} - \eta_{\alpha\beta} \to 0$ as $|\vec{x}| \to \infty$. This is a very weak condition. It's hard to imagine AF without this. It should be sufficient for my purposes in this talk.

If we have (M, g) is AF with U as above, a black hole is a non-empty connected component of $CI^-(U)$. The event horizon is the boundary of the black hole, $= \partial BH$. We will not be assuming any sort of C^1 continuation of the spacetime or similar.

Minkowski space is $\mathbb{R}^{3+1}$, with metric $\eta = -dt^2 + dr^2 + r^2 d\omega^2$ [$d\omega^2$ is the round metric on the sphere]. Let $u_\pm = \frac{1}{2}t \pm r$ and $U_\pm = \arctan(u_\pm)$. Then let $T = U_+ + U_-$ and $R = U_+ - U_-$. Then

$$\eta = (\cos^2 U_+ + \cos^2 U_-)(-dT^2 + dR^2 + \sin^2 R d\omega^2).$$

The front is a conformal factor, so it doesn't change any of the causal structure of the manifold. So, for many purposes we can drop the conformal factor. We can then embed this in $\mathbb{R} \times S^3$ and add points for future and null infinity $\mathcal{I}$, and spacelike infinity i^0 and future timelike infinity and past $i^\pm$. See fig 1.

The Schwarzschild metric is

$$g = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1} dr^2 + r^2 d\omega^2.$$

We can do similarly, $v = t + r + 2M \ln((r - 2M)/2M)$. Then the metric takes the form

$$g = -(1 - 2M/r)dv^2 + dvdr + r^2 d\omega^2$$

with $v \in \mathbb{R}$, $r \in (0, \infty)$, $\omega \in S^2$. We can do exfoliation by surfaces of constant v. If we have an outward null geodesic, (i.e. it increases in t and r, and then we change coordinates), then $dv/dr = 2/(1 - 2M/r)$ and so $2M/r$ becomes a barrier that can't be crossed. But see fig 2. This is an analytic continuation of the spacetime. We could do more [Kruskal extension]. Physically, this spacetime is a spherical body with mass M. Note that this now satisfies our condition for AF.

The Kerr black hole solution has parameters (M, a), where the mass is M, and a is some rotational parameter. The case $a = 0$ is Schwarzschild. These solutions are AF. It has a BH for $|a| \le M$. The horizon is now at $M + \sqrt{M^2 - a^2}$ which is largest root of Δ (see below). If $|a| < M$, then the exterior of the black hole is like Schwarzschild, but the interior is different. Let $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. The metric is then

$$g_{M,a} = -\left(\frac{r^2 - 2Mr + a^2\cos^2}{\Sigma}\right)dt^2 - \left(\frac{4aMr}{\Sigma}\right)\sin^2\theta d\phi dt + \cdots .$$

The coefficient of dt^2 is negative for some area outside of black holes, and so is spacelike somewhere outside the event horizon [the ergosphere]. Also, the cross term $d\phi dt$ says that dt is not orthogonal to slices of constant t, like it is for things like Schwarzschild.

2. $\mathbb{R}^{3+1}$ STABILITY

One of the basic problems of science is to predict the future from the present. We want to be able to do this in a continuous way, or else little errors would destroy all prediction. If we take Minkowski, and take some slice of it, we want to make sure it leads to an evolution more or less like the one we begin with, even with slight perturbations. We expect to always converge to stationary solutions, no matter the initial data, and so we should expect stationary solutions to be stable.

First was found hyperboloidal future stability by Friedrich in 1986. He found stability in the shaded region of Figure 3. The only dynamical part of the metric/curvature is the Weyl curvature. It is conformally invariant, so we expect that the Einstein equations are also.

Strategy: conformally transform the spacetime to a local problem in $\mathbb{R}\times S^3$ (i.e. extend a little outside of Minkowski space, and then local well posedness gives stability.) Not: The positive mass theorem implies that either you started with Minkowski, or the data has mass at infinity. So you have $1-2M/r$ terms showing up, which we then get $= 1 - 2M|\sin(R-\pi)|$, and so we have a cusp singularity at $R = \pi$. Thus our solution can't extend over i^0, and so this argument fails if the slice goes through that point, and so it only works for hyperboloidal.

Global stability by Christodoulou and Klainerman in 1993. For data goes out to i^0, and rougher, so that maybe it doesn't allow C^1 conformal compactification. Then K-Nicolo got better results, and Zipser did it with EM. Lindblad-Rodnianski used different gauge conditions, and divergences from that could be bounded, which simplified the argument. Bieri did similar with weaker ideas of AF.

Theorem 2.1. *There is some small distance $\epsilon > 0$ for $(\Sigma, \bar{g}, \bar{k})$. Let $\mathrm{tr}K = 0$, $\bar{g} = (1 - 2M/r)\delta_{ij} + o_4(r^{-3/2})$ and $\bar{k}_{ij} = o_3(r^{-5/2})$. Let $x_0 \in \Sigma$, and d_0 is $\bar{g}$ distance in Σ from x_0. Let $B_{ij} = \epsilon_j^{ab}\nabla_a(R_{ib} - \frac{1}{4}g_{ib}R)$ (Bach tensor?). Suppose*

$$\epsilon > J_0 = \sup_\Sigma((d_0^2+1)^3|Ric[\bar{g}]|^2)+\int_\Sigma\left(\sum_{l=0}^{3}(d_0^2+1)^{l+1}|\nabla^l\bar{k}|^2\sum_{l=0}^{1}(d_0^2+1)^{l+3}|\nabla^l B|^2\right)$$

is sufficiently small. Then the maximal globally hyperbolic development is geodesically complete and Riem $\to 0$ *along any geodesic.*

Strategy: used generalized energies, and then improves them by using symmetries. Used good foliation. Einstein equation is a non-linear PDE, and then used good properties of non-linearity. First two can be understood in terms of linear fields.

Also, we should look at simpler model equations, and then look at the full equations. Models: null geodesic equation:

$$\dot{\gamma}^\alpha \nabla_\alpha \dot{\gamma}^\beta = 0$$

Wave :

$$\nabla^\alpha \nabla_\alpha u = 0$$

Maxwell:

$$\nabla^\alpha F_{\alpha\beta} = 0 = \nabla_{[\alpha} F_{\alpha\beta]}$$

Weyl: For a 4 index tensor that has same symmetries of Weyl tensor:

$$\nabla_{[\epsilon} W_{\alpha\beta]\gamma\delta} = 0$$

The linearization of Einstein in Minkowski space is this last one. Outside of BH we get extra terms from background curvature, so get plus some coupling terms. The strategy was to look at these and get good decay, and use that decay to get good decay of Weyl.

3. Kerr stability conjecture

He'd never seen a rigorous statement of this, so he won't try. He doesn't expect a rigorous statement to come up till it's been proven, since the details will be tricky. But the idea is that we want that if we take initial data like Kerr black hole data, they will evolve to one close to Kerr.

Data: We expect $|a| < M$, so that we don't have critical spin. [Funny things happen at maximal spin...] We specify initial data on a hypersurface Σ, either out to future null infinity or spacelike infinity. It doesn't matter which one in this case, because the initial data will be asymptotically Schwarzschildean, and it is known how to change between the two asymptotic conditions. We need Σ to extend a little inside the event horizon, in case the horizon changes some after perturbation.

Solutions: It should have an AF region in the future, and should keep the black hole. It should converge to *some* Kerr solution at timelike infinity, i^+, i.e. $g \to g_{M',a'}$, with $|M - M'| + |a - a'|$ small.

Birkhoff's theorem says that any spherically symmetric, AF solution of the Einstein equations, is exactly the Schwarzschild solution. Thus there is no Schwarzschild stability, since can't do perturbations. general non-spherical perturbations will provide some angular momentum and thus the solution will not be in the Schwarzschild class anymore. However, we do expect $|a| << M$ Kerr solutions are stable.

Obstacles for proving decay for model equations: We want to show decay to 0 outside of black hole.

(1) Orbits. There are null geodesics that fail to go to infinity or fall into BH, for instance. These obviously then don't decay. But similar things can happen for the wave equation, for instance.

Given $\epsilon_1, \epsilon_2, T > 0$ and null geodesic γ, then there exists solutions of the wave equation $\nabla^\alpha \nabla_\alpha u = 0$ with supp $[u(0)] \subset \gamma(0) + B_{\epsilon_1}$ where for some $t \in [0, T]$ we can find the solution such that

$$\int_{\{t\}\times(\gamma(t)+B_{\epsilon_1})} |\partial u|^2 > (1-\epsilon_2) \int_{\{t=0\}} |\partial u|^2.$$

This is a result of Ralston. So, most of the energy can be made to stay near the geodesic. This prevents the decay one wants, but there are tricks to get around which will be discussed in the next lecture. Eventually, though, this type of data will decay.

(2) For $a \neq 0$, there is no positive, conserved energy.

(3) For the Maxwell equations, and for the linearized Einstein equations, there are bound states (by adding charges for Maxwell, or changing of mass for linearized Einstein.) These problems don't have the "will eventually decay away" property like the wave equation solutions in 1.

4. Energy generation and strengthening

This is sometimes called the vector field method.

There are some common properties of T, the stress energy tensor:

(1) $T_{\alpha\beta} = T_{\beta\alpha}$ (symmetry),

(2) For X, Y causal future directed vector fields, $T_{\alpha\beta} X^\alpha Y^\beta \geq 0$ (dominant energy condition),

(3) $\nabla^\alpha T_{\alpha\beta} = 0$ (divergence free), and

(4) $T_\alpha^\alpha = 0$ (trace free).

Some of these might fail in the model equations, for some analogue of the stress-energy tensor.

Null geodesics: $\dot\gamma^\alpha \dot\gamma^\beta$ satisfies 1,2, some analog of 3 and 4.

Wave: $\partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial_\gamma u \partial^\gamma u$ satisfies 1 through 3.

Maxwell: $(F_\alpha^\mu F_\beta^\nu - (*F)_\alpha^\mu (*F_\beta^\nu) g_{\mu\nu}$ satisfies all of them

Weyl: We have the Bel-Robinson tensor,

$$\left(W_\alpha{}^\mu{}_\gamma{}^\sigma W_\beta{}^\nu{}_\delta{}^\tau + (*W)_\alpha{}^\mu{}_\gamma{}^\sigma (*W)_\beta{}^\nu{}_\delta{}^\tau\right) g_{\mu\nu} g_{\sigma\tau}$$

and satisfies analogs of 1 through 4.