Dafermos

Fig 1.

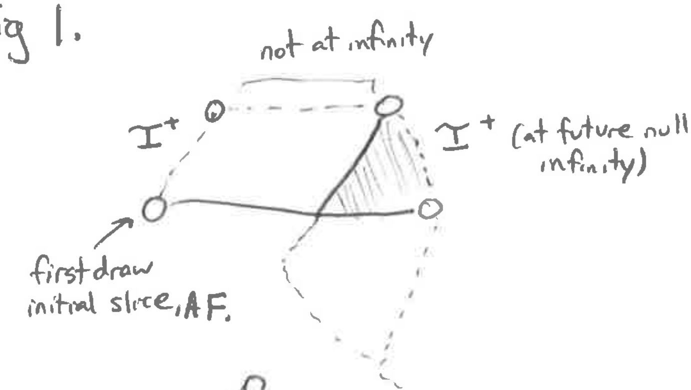

Fig 2

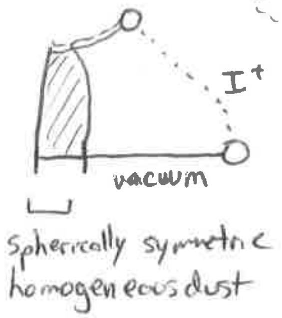

Fig 3

THE COSMIC CENSORSHIP CONJECTURES

MIHALIS DAFERMOS

Slide 5: The fact that vacuum has dynamics in GR is a completely new thing over Newtonian physics.

The proof is non-constructive because it uses Zorn's lemma. This is unusual because we claim uniqueness. Fortunately J. Sbierski has proven it recently without using Zorn's lemma, completely constructively!

Slide 10: Penrose representation of the manifold is in Figure 1. "Impeccable" initial data should have any sort of noncompactness should be associated with nice ends. With one end, the only one is Minkowski. So, it must have 2 ends. The shaded region in Figure 1 is the part mentioned in the Schwarzschild metric for $r > 2M$.

Slide 11: See Figure 2. Matter is contained in the shaded area, but the rest is Schwarzschild. We don't want to worry about incompleteness when we think about vacuum solutions, so we think of the two-ended data anyway.

Slide 13: The stated theorem is a corollary of the full Penrose inequality theorem, since the conditions of that theorem are trivially satisfied by the Schwarzschild metric.

Slide 17: If we wiggle the initial data of Minkowski (see Fig 3), we have the stability and the complete null infinity. A standard result says that a trapped surface can never be in the past of future null infinity.

Also, there is not known examples that fail this conjecture without genericity, but we conjecture and have good reason to believe there is.

Slide 18: Infinite tidal forces mean that Jacobi fields on an observer are infinitely twisted up.

Slide 19: The Cauchy horizon is the boundary of the original maximal development. Kerr has the worst problems, since every incomplete observer in the original spacetime survives in some other dimension that is not dependent on the initial data!

Slide 20: In the Kerr family itself, it is generic that strong cosmic censorship fails, since just the Schwarzschild family obeys it.

Why is it called cosmic censorship? We could think of the weak conjecture by saying something like "any incompleteness has to fall into black holes". In this case, incompleteness is "censored." But this statement could be logically independent from how we formulated the problem.

The cosmic censorship conjectures

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MSRI, September 10, 2013

Outline

1. Brief review of the Cauchy problem (Dan's talk)
2. The Schwarzschild solution and Penrose's incompleteness theorem
3. The cosmic censorship conjectures

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4. What's the status? (**part 2**)

1. Brief review of the Cauchy problem

The ***cosmic censorship conjectures*** can be viewed as a way to come to term with "singularities" in general relativity.

The issue of "singularities" and their interpretation plagued the theory from the very early years, causing immense–*embarassing, in retrospect!*–confusion.

Part of this confusion simply stemmed from poor understanding of Lorentzian geometry. But an additional fundamental element was failure to understand that singularity is a *dynamical* prediction of the theory, i.e. "singular" behaviour can arise dyamically from initial states which are not singular.

Of course, to make sense of this, one needs a well-defined notion of **dynamics**. The existence of dynamics is in turn intimately connected to the essential hyperbolic nature of the Einstein equations, allowing for a *well-posed Cauchy problem*. The formulation and proof of this result was an important conceptual achievement that should not be overlooked!

Let us review the existence of dynamics from Dan's talk.

The fundamental theorem of general relativity

Theorem 1 (Choquet-Bruhat 1952, Choquet-Bruhat–Geroch 1969)**.**
Let $(\Sigma, \bar{g}, K)$ *be a smooth vacuum initial data set. There exists a unique smooth spacetime* $(\mathcal{M}, g)$ *such that*

1. $\mathrm{Ric}(g) = 0$
2. $(\mathcal{M}, g)$ *is globally hyperbolic with Cauchy surface* Σ*, with induced first and second fundamental form* $\bar{g}$*,* K *respectively*
3. *Any other smooth spacetime with properties 1., 2., isometrically embeds into* $\mathcal{M}$*.*

Similar theorems can be proven for suitable coupled Einstein–matter systems.

General (meta)remarks on the Cauchy problem

The above theorem says that there exists an unambiguous notion of dynamics.

It allows us to rationally entertain whether a certain property of solutions is ***stable***, in the sense that the property persists on perturbation of initial data, or whether a certain property is ***generic*** in the sense that the property holds for all solutions arising from an open and dense set in the moduli space of initial data.

There are of course settings where one has to be careful in applying these notions: For instance, when solving *backwards* in time, where knowledge of initial data is often secretly correlated with particular properties of the past evolution. This is the case, for instance, in **cosmology**, to which I defer to the talk of Hans Ringström.

This talk will focus on *future* evolution of *isolated* self-gravitating systems.

Asymptotically flat data sets (review from Dan's talk)

For pedagogical clarity, let us restrict to the vacuum.

The mathematical idealisation of the notion of the initial state of an "isolated gravitational system" is given by an *asymptotically flat* data set $(\Sigma, \bar{g}, K)$.

This is defined to mean that there exists a compact set C such that $\Sigma \setminus C$ is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a closed ball, such that moreover the metric $\bar{g}_{ij}$ and 2nd fundamental from K_{ij} approach the Euclidean metric and 0, respectively as $|x| \to \infty$, at suitable rates.

Far-away observers not part of the system–***typically, us!***–must then be understood as belonging to an asymptotic boundary of spacetime "at infinity".

2. The Schwarzschild solution and Penrose's incompleteness theorem

The Schwarzschild metric

The *Schwarzschild solution* was the first non-trivial explicit solution of the Einstein vacuum equations to be discovered, already in December 1915.

In local coordinates, its metric element is most succinctly expressed as

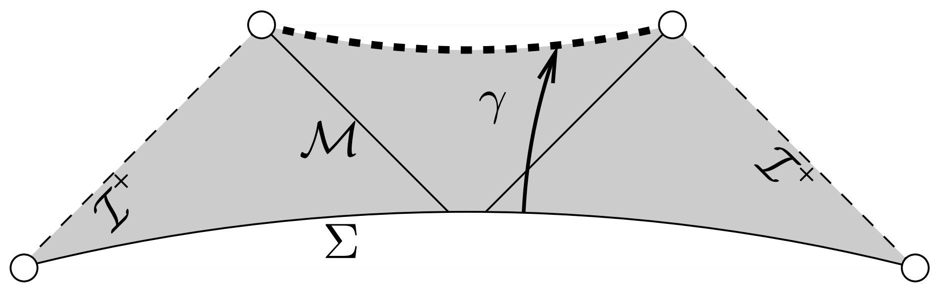
$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2),$$

where M is a real parameter.

Much confusion was caused as to what ambient manifold $\mathcal{M}$ the above metric expression "should" live on.

Schwarzschild as a future Cauchy evolution

From the dynamical point of view, it makes sense to consider Schwarzschild as the **maximal future Cauchy development** of spherically symmetric asymptotically flat vacuum initial data defined on Σ.

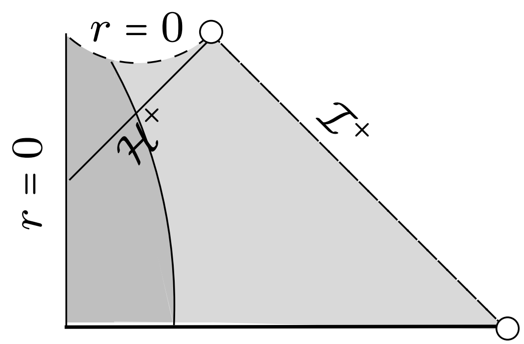


For this, however, we must generalise the notion of asymptotic flatness to allow for ***two*** asymptotically flat ends, for only this case is allowed by the spherically symmetric *vacuum*.

Of course, two-ended data *as such* is not physically relevant. Why does one even consider this then?

Schwarzschild as a proxy for Oppenheimer–Snyder

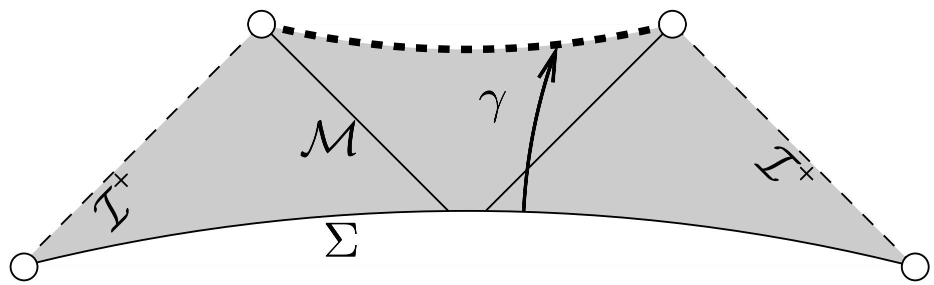
If we consider the maximal Cauchy development of an initially homogeneous dust ball surrounded by vacuum (as a solution of the coupled **Einstein–dust** system), one obtains a spacetime (the celebrated *Oppenheimer–Snyder*) that will coincide with Schwarzschild outside the support of the matter.



It will be *pedagogically* useful, however, for the purpose of this talk, to treat two-ended Schwarzschild as a sort of "proxy" for Oppenheimer–Snyder.

Thus, please *suspend disbelief* and treat the two-ended data as physically admissible!

Schwarzschild is geodesically incomplete!



Though a maximal Cauchy development of perfectly admissible (in view of the previous!) asymptotically flat data, the Schwarzschild spacetime turns out to be

future causally geodesically incomplete.

That is to say, **there exist observers–*like poor* γ *depicted above*–who live for only finite proper time.**

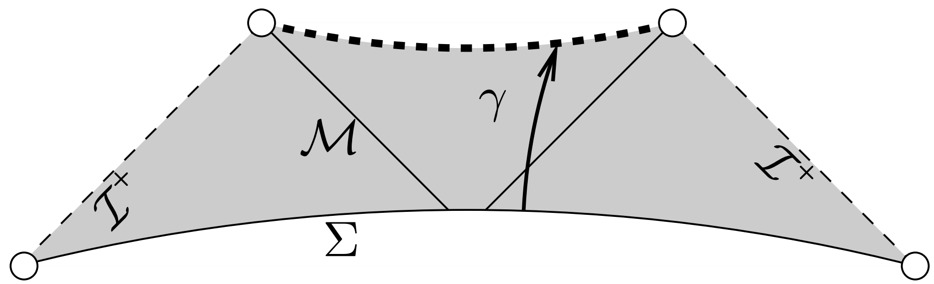
To the extent that this was even understood, this was originally widely thought to be a pathology due to the high degree of symmetry of the data, that would certainly go away upon perturbation of initial data on Σ.

PENROSE's incompleteness theorem

This expectation was falsified spectacularly with the following corollary of PENROSE's celebrated incompleteness theorem, which says that Schwarzschild's geodesic incompleteness is **stable** to perturbation of initial data:

Theorem (PENROSE 1965)**.** *For all small perturbations of Schwarzschild initial data on Σ, the maximal vacuum Cauchy development is still geodesically incomplete.*

Back to Schwarzschild



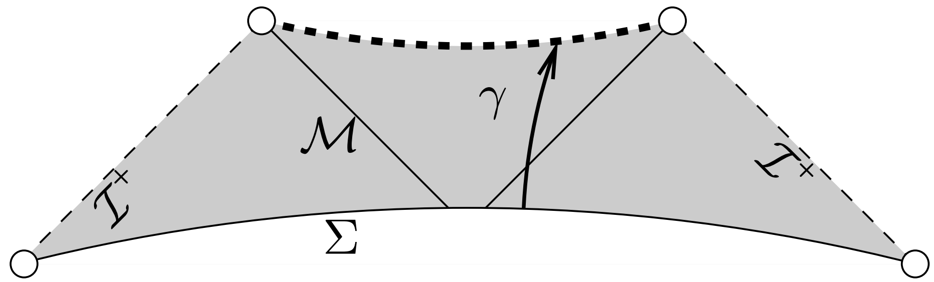
Since incompleteness occurs for admissible data and turns out to be a stable property in the theory, then, *like it or not*, one has to come to terms with it.

The so-called *cosmic censorship conjectures* are best viewed as a way to try to come to terms with Schwarzschild's incompleteness: They were originally motivated by two "good" aspects associated to the nature of the Schwarzschild geometry, which make its incompleteness seem more "acceptable".

We turn to discuss these.

3. The cosmic censorship conjectures

Null infinity and the black hole notion



Null infinity, what we have been depicting by $\mathcal{I}^+$, is the asymptotic boundary "at infinity" idealising "far-away observers *in the wave zone*", i.e. observers who would receive gravitational and electromagnetic radiation from perturbations of the system.

The first property that we notice about Schwarzschild is that the past of future null infinity $J^-(\mathcal{I}^+)$ has a **non-trivial complement**, the so-called ***black hole*** region, and all incomplete γ must enter this region.

Thus, far away observers in the wave zone do not "see" incompleteness.

Moreover, null infinity $\mathcal{I}^+$ itself is *complete*. That is to say, though poor γ only lives for a finite time, LIGO can observe for "infinite" time.

Weak cosmic censorship

The *weak cosmic censorship conjecture* is the statement that this completeness of null infinity is **generic**:

Conjecture (Weak cosmic censorship, Penrose 1969)**.** *For generic asymptotically flat vacuum initial data, the maximal vacuum Cauchy development has a complete null infinity.*

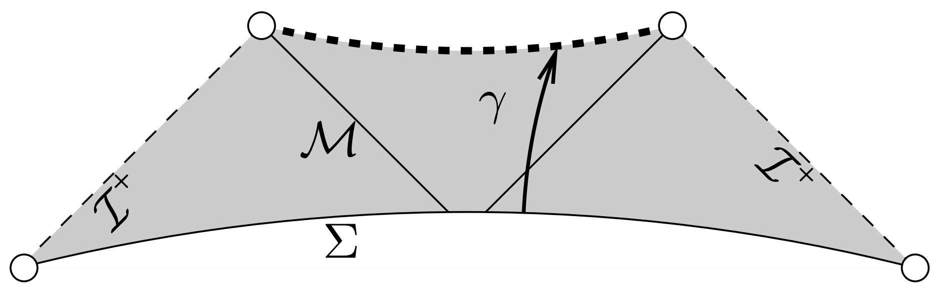
Again, for pedagogical clarity I have restricted to the vacuum. This formulation is due to Christodoulou.

Why the assumption of genericity? Why not conjecture this for all solutions? See **part 2**. . .

What happens to incomplete observers?

There is another “good” aspect of Schwarzschild’s incompleteness, *although it might at first not be so clear why this should be considered to be good:*

All observers γ which live for only finite time encounter *infinite* curvature, in fact they are **torn apart** by *infinite* tidal forces.

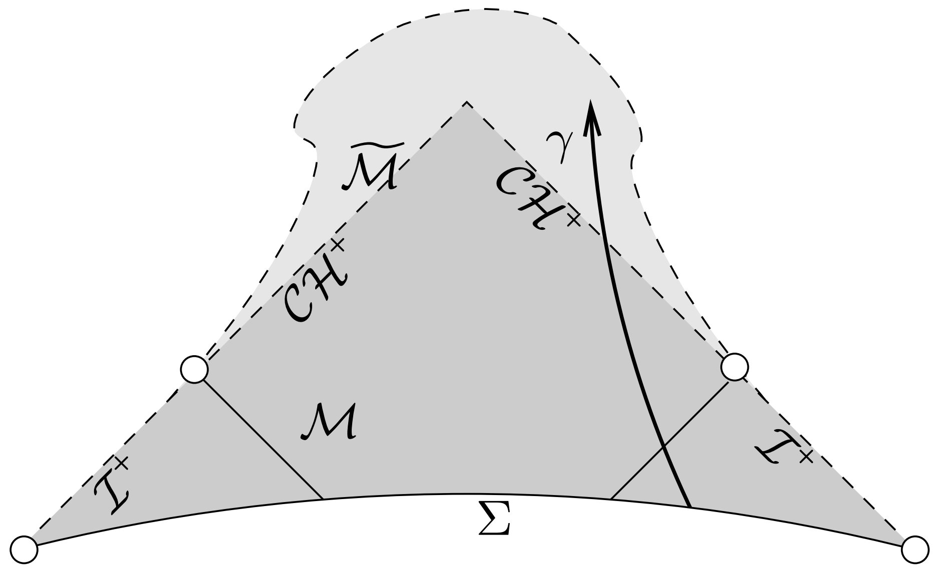


In what sense is this “good” ????

One can at least be satisfied that this incompleteness brings with it a sense of “closure”. Being torn apart might not be so pleasant for them, but everyone is “accounted for” by the classical theory! We know exactly what happens.

Kerr $0 < |a| < M$

The Schwarzschild family is in fact a 1-parameter subfamily of the 2-parameter *Kerr* family of vacuum solutions.



The maximal Cauchy development of Kerr initial data is future extendible (smoothly!) so that **all** incomplete geodesics γ fall into the extension. Observer γ is thus *not* destroyed, but the extension (which is not *globally hyperbolic*) is not determined by initial data, so the future of γ cannot be predicted.

Strong cosmic censorship

Conjecture (Strong cosmic censorship, PENROSE 1972)**.** *For generic asymptotically flat vacuum initial data, the maximal Cauchy development is future inextendible as a suitably regular Lorentzian manifold.*

One should think of this conjecture as a statement of global uniqueness, or **determinism**.

As opposed to the weak cosmic censorship conjecture, here it is clear *a priori* why the assumption of genericity is required, as we have just seen that the celebrated Kerr family of vacuum solutions fail to satisfy the above statement in a spectacular way.

We will see in **part 2** why, *despite the existence of the Kerr family*, one can still reasonably reasonably hope that the above conjecture is true.