An introduction to the Penrose inequality conjecture

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Outline

1. A look at the Kerr-Newman spacetimes.
2. The Penrose inequality for black holes.
3. The Penrose inequality conjecture for initial data sets.
4. The Riemannian Penrose inequality conjecture.
5. The Bray and Khuri approach to the general Penrose inequality.

- The family of Kerr-Newman spacetimes is fundamental in General Relativity. Each element is identified by three real numbers m, a and q.
- The global properties of the spacetime depend on these values.
- Here, sufficient to restrict each spacetime to a suitable open subset and define:

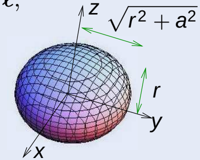
Definition (Kerr-Newman spacetime)

For $m, a, q \in \mathbb{R}$ let $M_a = \mathbb{R} \times \left(\mathbb{R}^3 \setminus \{x^2 + y^2 \leq a^2, z = 0\}\right)$, with (x, y, z) Cartesian coordinates in $\mathbb{R}^3$. The Kerr-Newman spacetime of mass m, specific angular momentum a and charge q is the spacetime $(M_a, g_{m,a,q})$ where

$$g_{m,a,q} = \underbrace{-dt^2 + dx^2 + dy^2 + dz^2}_{\eta} + \frac{r^2\left(2mr - q^2\right)}{r^4 + a^2 z^2} \ell \otimes \ell,$$

where $r \in C^\infty(M_a, \mathbb{R}^+)$ is defined by $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$ and

$$\ell = dt + \frac{r}{r^2 + a^2}(xdx + ydy) + \frac{a}{r^2 + a^2}(ydx - xdy) + \frac{zdz}{r}.$$



Some properties:

- Let $R := \sqrt{x^2 + y^2 + z^2}$. At large R, the metric is $g_{m,a,q} = \eta + O(\frac{1}{R})$

- The hypersurface $\mathcal{H}_{r_0} = \{r = r_0\}$ ($r_0 > 0$ const.) is topologically $\mathbb{R} \times \mathbb{S}^2$.
- $\mathcal{H}_{r_0}$ is null for some $r_0 > 0$ iff $\sqrt{a^2 + q^2} \leq m \neq 0$.

Kerr-Newman black hole spacetime: Kerr-Newman spacetime with $\sqrt{a^2 + q^2} \leq m \neq 0$.

- The null hypersurface is $\mathcal{H}_{r_+}$, where $r_+ := m + \sqrt{m^2 - a^2 - q^2} > 0$.

Given any $p \in M_a$:

- If $r(p) > r_+$ then, for any $R_0 > 0$, there exists a future directed causal curve starting at p and entering the region $\{R > R_0\}$.
- If $r(p) < r_+$, then all future directed causal curves starting at p lie in $\{r < r_+\} \longrightarrow$

Signals cannot "escape" to infinity.

$\mathcal{H}_{r_+}$ separates both behaviours. Defines the event horizon of the black hole.

- All sections S in $\mathcal{H}_{r_+} = \mathbb{R} \times \mathbb{S}^2$ are isometric to each other and have area

$$|S| = 8\pi m \left(m + \sqrt{m^2 - a^2 - q^2}\right) - 4\pi q^2 \leq 16\pi m^2.$$

This is the basic inequality behind the Penrose inequality conjecture.

- Equality iff $a = q = 0$. This is the Schwarzschild class of spacetimes:

$$\left(M_0 = \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}), g_m = \eta + \frac{2m}{R}(dt + dR)^2\right), \quad m \in \mathbb{R}^+, \quad R = |x|_\delta$$

Black hole spacetimes

The notion black hole requires a notion of infinity. A natural one is asymptotic flatness.

- Several (inequivalent) definitions of asymptotic flatness. For definiteness:

Definition

A 4-dimensional spacetime $(M, g^{(4)})$ is asymptotically flat if it admits an asymptotically flat 4-end, i.e.

- *An open submanifold $M^\infty \simeq \mathbb{R} \times (\mathbb{R}^3 \setminus \overline{B}(R_0))$ such that*
- *$\exists C > 0$ such that the components $g_{\mu\nu}$ of $g^{(4)}|_{M^\infty}$ in Cartesian coordinates (t, x, y, z) satisfy (with $R = \sqrt{x^2 + y^2 + x^2}$)*

$$|g_{\mu\nu}| + |g^{\mu\nu}| + R|g_{\mu\nu} - \eta_{\mu\nu}| + R^2|\partial_\sigma g_{\mu\nu}| + R^3|\partial_\sigma \partial_\rho g_{\mu\nu}| \leq C.$$

Any Kerr-Newman spacetime is asymptotically flat.

- For any $r > R_0$ define $M_r = \{p \in M^\infty : R(p) > r\}$.
- The black hole region (w.r.t. the asymptotically flat four-end) is

$$\mathcal{B} := \{p \in M;\ \exists r(p) > R_0 \text{ such that all future directed causal curves starting at } p \text{ lie in } M \setminus M_{r(p)}\}.$$

- if $\mathcal{B}$ is non-empty, $(M, g^{(4)})$ is a black hole spacetime.

The event horizon $\mathcal{H}$ is the topological boundary of $\mathcal{B}$.

- $\mathcal{H}$ is a Lipschitz null hypersurface ruled by future inextendible null geodesics.

Recall:

Definition (Dominant and null convergence conditions)

If for all $k_1, k_2 \in T_pM$ null and future directed and all $p \in M$:

- *$Ein(k_1, k_2) \geq 0$ then $(M, g^{(4)})$ satisfies the dominant energy condition (DEC).*
- *$Ein(k_1, k_1) \geq 0$ then $(M, g^{(4)})$ satisfies the null energy condition (NEC).*

A hypersurface Σ is achronal if no two distinct points in Σ can be joined by a timelike curve.

- An important result concerning event horizons is the Area Theorem.

Theorem (Hawking '72, Chruściel, Delay, Galloway, Howard '01)

Assume $(M, g^{(4)})$ is a black hole spacetime satisfying the NEC. Let Σ_1 and Σ_2 be achronal, spacelike hypersurfaces and define $\mathcal{H}_{\Sigma_a} := \mathcal{H} \cap \Sigma_a$ $(a = 1, 2)$. If every point p in $\mathcal{H}_{\Sigma_1}$ can be joined to $\mathcal{H}_{\Sigma_2}$ by a future directed curve starting at p then $|\mathcal{H}_{\Sigma_1}| \leq |\mathcal{H}_{\Sigma_2}|$.

- Any such $\mathcal{H}_\Sigma$ is a section of the event horizon.

Heuristics behind the Penrose inequality conjecture

- In physical terms, it is expected that any black hole spacetime must settle down to an asymptotic stationary state in the distant future.
- All matter fields (except electromagnetic ones) are expected to be radiated away or fall into the black hole region, so the asymptotic spacetime will be electrovacuum.
- Black hole uniqueness theorem: Under suitable conditions, a stationary electrovacuum black hole spacetime must be isometric to a Kerr-Newman black hole outside their event horizons.

Consequence:

Expectation

For any black hole spacetime, $\exists\, m, a, q \in \mathbb{R}$ satisfying $\sqrt{a^2 + q^2} \leq m \neq 0$ such that the asymptotically flat four-end of the black hole approaches (in a suitable sense) $(M_{r_+}, g_{m,a,q})$ when $t \to +\infty$.

- Asymptotically flat 4-ends admit a notion of total energy-momentum vector P.
- P is an element of an abstractly defined Lorentzian vector space (V, η).

Fundamental property:

- An asymptotically 4-end satisfying the DEC and approaching (in a suitable sense) a Kerr-Newman spacetime $(M_a, g_{m,a,q})$ satisfies $M_{ADM}^2 := -|P|_\eta^2 \geq m^2$.
- Physically: gravitational radiation can only extract energy from the spacetime.

The first key insight by Penrose is the following chain of inequalities:

- Let $\mathcal{H}_\Sigma$ be any section of the event horizon and $\mathcal{H}_{\Sigma_\infty}$ a section of the event horizon in the asymptotic future:

$$|\mathcal{H}_\Sigma| \leq |\mathcal{H}_{\Sigma_\infty}| \leq 16\pi m^2 \leq 16\pi M_{ADM}^2$$

Area theorem — Property of Kerr-Newman — Energy can only be radiated away

- The resulting inequality $|\mathcal{H}_\Sigma| \leq 16\pi M_{ADM}^2$ involves no future asymptotic properties.
- However, still involves the event horizon, which is a global concept in the spacetime.

The second key insight of Penrose was to argue that a similar type inequality (involving total ADM mass and area of suitable surfaces) should hold for a general class of initial data sets.

Initial data sets

- Initial data set: Triple (Σ^n, g, K), $(n \geq 3)$: (Σ, g) Riemannian manifold (possibly with boundary) and K symmetric 2-cov. tensor.
- Dominant Energy Condition (DEC): $\rho \geq |\boldsymbol{J}|_g$ where

$$16\pi\rho := R_g - |K|_g^2 + k^2, \qquad 8\pi\boldsymbol{J} := \mathrm{div}_g(K - k\,g) \qquad k = \mathrm{tr}_g K\,.$$

- (Σ, g, K) is asymptotically flat if $\Sigma = \mathcal{K} \cup (\mathbb{R}^n \setminus \overline{B}(R_0))$, $\mathcal{K}$ compact and in Cartesian coordinates in $\mathbb{R}^n \setminus \overline{B}(R_0)$:

$$g_{ij} - \delta_{ij} = O_{(2)}\Big(\frac{1}{R^p}\Big), \quad K_{ij} = O_{(1)}\Big(\frac{1}{R^{p+1}}\Big), \quad \rho, |J|_g = O\Big(\frac{1}{R^q}\Big), \quad p > \frac{n-2}{2}, q > n.$$

- ADM-energy E_{ADM} and ADM-linear momentum P_{ADM}: let $c_n := \frac{1}{2(n-1)\omega_{n-1}}$,

$$E_{ADM} := c_n \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\,\nu^i dS_r\,, \quad P_{i\,ADM} := 2c_n \lim_{r\to\infty} \int_{S_r} (K_{ij} - g_{ij}k)\,\nu^j dS_r\,.$$

- Positive Mass Theorem: [Schoen & Yau ’79, Eichmair, Huang, Lee, Schoen ’11] Under DEC and in dimensions $3 \leq n \leq 7$: $E_{ADM}^2 - |P_{ADM}|_\delta^2 \geq 0$.
- When (Σ, g, K) is embedded in a spacetime $(M^{n+1}, g^{(n+1)})$: the energy-momentum vector P has components (E_{ADM}, P_{ADM}) and $M_{ADM}^2 = E_{ADM}^2 - |P_{ADM}|_\delta^2$.

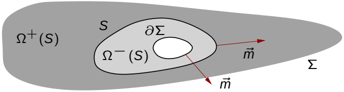
Weakly outer trapped surfaces and future trapped region

The Penrose inequality involves surfaces that, in an appropriate sense, replace sections of the event horizon.

- **Surface:** smooth closed codimension-one embedded submanifold of $\mathrm{int}(\Sigma)$.
- A surface is **bounding** if $\Sigma \setminus S$ has more than one connected component.
- Exterior region $\Omega^+(S)$: the unbounded connected component of $\Sigma \setminus S$.

S_1 encloses S_2 if $\Omega^+(S_2) \subset \Omega^+(S_1)$.

- $\Omega^- = \Sigma \setminus \overline{\Omega^+}$: interior region.



- Mean curvature H_S always computed with respect to the normal m pointing towards $\Omega^+(S)$.

Definition

A bounding surface S is weakly outer trapped if $\theta_+ := H_S + tr_S K \leq 0$. and a marginally outer trapped surface (MOTS) if $\theta_+ = 0$.

The future trapped region $\mathcal{T}^+_\Sigma \subset \Sigma$ is the union of the interior domains of all weakly outer trapped surfaces in Σ.

Theorem ([Andersson & Metzger '07], [Eichmair '09])

Let (Σ, g, K) be asymptotically flat and of dimension $3 \leq n \leq 7$. Then the topological boundary $\partial\mathcal{T}_\Sigma^+$ is either empty or a MOTS.

- In any black hole spacetime, the future trapped region always lies inside the black hole region.

Second key insight by Penrose: Assume the weak cosmic censorship hypothesis holds

"Generic" 3-dimensional asymptotically flat initial data sets for "reasonable" matter models and admitting a weakly outer trapped surface can be embedded as an achronal hypersurface in a black hole spacetime.

- So, if weak cosmic censorship holds then $\mathcal{H}_\Sigma$ exists and encloses $\partial\mathcal{T}_\Sigma^+$.
- $\mathcal{H}_\Sigma$ cannot be located directly from the initial data, but it necessarily must have at least as much area as the minimal area needed to enclose $\partial\mathcal{T}_\Sigma^+$ [Jang & Wald '77].

The Penrose inequality for initial data sets

- For any bounding S: $|S_{\min}(S)|$ infimum of areas of all surfaces enclosing S.

$$|\mathcal{H}_\Sigma| \geq |S_{\min}(\partial \mathcal{T}_\Sigma^+)|.$$

- Combining with the Penrose inequality for asymptotically stationary black holes:

$$16\pi M_{ADM}^2 \geq |\mathcal{H}_\Sigma|.$$

Conjecture (Penrose inequality)

Let (Σ, g, K) be a 3-dimensional asymptotically flat initial data set satisfying DEC. Then

$$16\pi M_{ADM}^2 \geq |S_{\min}(\partial \mathcal{T}_\Sigma^+)|.$$

Moreover, if equality holds then $(\Sigma \setminus \mathcal{T}_\Sigma^+, g, K)$ can be isometrically embedded into the Schwarzschild spacetime.

Two basic ingredients support the inequality:

- Expected behaviour of black hole spacetimes.
- The validity weak cosmic censorship conjecture.

Important problem to either prove the conjecture or find counterexamples.

- In particular, it would provide a strengthening of the positive mass theorem for initial data sets with appropriate (marginally) outer trapped boundary.

Alternative versions

- Equality case is only conjectured to imply embeddedness in Schwarzschild.
- Must be so, because not all initial data sets of Schwarzschild satisfy equality.
- Furthermore, the statement of the conjecture is not invariant under $K \to -K$.

These two properties have led to some proposals generalizing the conjecture.

- In several cases, counterexamples have been found.

There is, however, one version for which:

- All slices of Schwarzschild satisfy equality.
- Symmetric under $K \to -K$.
- No counterexamples are known so far.

Define $\mathcal{T}_\Sigma^-$ as the future trapped region of $(\Sigma, g, -K)$.

Question

Under the same assumptions as in the Penrose inequality conjecture, is the following inequality true?

$$16\pi M_{ADM}^2 \geq |\partial(\mathcal{T}_\Sigma^+ \cup \mathcal{T}_\Sigma^-)|.$$

Stronger than the standard Penrose inequality. Not supported by the heuristics above.

- However, (a version of) it holds in the spherically symmetric case.

The Penrose inequality in the spherically symmetric case

- (Σ, g, K) is spherically symmetric if the group $SO(3)$ acts by isometries on (Σ, g) with orbits diffeomorphic to $\mathbb{S}^2$ or points, and leaves K invariant.

Theorem (Malec & O'Murchadha '94, Hayward '96)

Let (Σ, g, K) be 3-dimensional, asymptotically flat, satisfying DEC and spherically symmetric. Then $16\pi E_{ADM}^2 \geq |\partial(\mathcal{T}_\Sigma^+ \cup \mathcal{T}_\Sigma^-)|$.

- Since $E_{ADM}^2 \geq M_{ADM}^2$ this is weaker than the Penrose inequality conjecture.
- Interesting problem: prove $16\pi M_{ADM}^2 \geq |\partial(\mathcal{T}_\Sigma^+ \cup \mathcal{T}_\Sigma^-)|$ in the spherical case.

The proof uses the Hawking mass: S orientable surface in (Σ^3, g, K)

$$M_H(S) = \sqrt{\frac{|S|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_S \left(H_S^2 - (\mathrm{tr}_S K)^2\right) dS\right).$$

Key properties:

- $M_H(\partial \mathcal{T}_\Sigma^\pm) = \sqrt{|\mathcal{T}_\Sigma^\pm|/16\pi}$ (does not require spherical symmetry).
- Let S_r be the SO(3) orbit of area $4\pi r^2$ outside $\mathcal{T}_\Sigma^+ \cup \mathcal{T}_\Sigma^-$. Then $M_H(S_r)$ is monotonically increasing in r.
- $\lim_{r\to\infty} M_H(S_r) = E_{ADM}$.

Since either $\mathcal{T}_\Sigma^+$ encloses $\mathcal{T}_\Sigma^-$ or viceversa, these three properties prove the theorem.

Riemannian Penrose inequality

A particularly important case of the Penrose inequality involves time symmetric initial data sets $(\Sigma, g, K = 0)$.

- $\mathcal{T}_\Sigma^+ = \mathcal{T}_\Sigma^-$ and $\partial\mathcal{T}_\Sigma^+$: outermost closed minimal surface in (Σ, g). $\longrightarrow$ Hence its own minimal area enclosure.
- The DEC becomes $R_g \geq 0$.
- The total ADM mass coincides with the total ADM energy $M_{ADM} = E_{ADM}$.

So, the conjecture involves

- Asymptotically euclidean 3-dim Riemannian manifolds (Σ, g) of non-negative scalar curvature $R_g \geq 0$ with outermost minimal (compact) boundary $\partial\Sigma$.
- The inequality reads

$$16\pi M_{ADM}^2 \geq |\partial\Sigma|$$

- The equality case is the Schwarzschild space of mass $m = M_{ADM} > 0$:

$$\Sigma = \mathbb{R}^3 \setminus B(m/2), \qquad g_{Sch} = \left(1 + \frac{m}{2|x|_\delta}\right)^4 \delta$$

- The heuristic argument for the Penrose inequality is specifically $3+1$-dimensional.
- However, the statement of the inequality can be easily extended to any dimension.

The Schwarzschild space has an immediate generalization: Schwarzschild *n*-space

$$\Sigma = \mathbb{R}^n \setminus B((m/2)^{1/(n-2)}), \qquad g_{Sch} = \left(1 + \frac{m}{2|x|_\delta^{n-2}}\right)^{\frac{4}{n-2}} \delta$$

Conjecture (Riemannian Penrose inequality in arbitrary dimension)

Let (Σ, g) be an n-dim, $n \geq 3$ asymptotically flat Riemannian manifold with outermost minimal boundary $\partial\Sigma$ and satisfying $R_g \geq 0$. Then, its ADM mass M_{ADM} satisfies

$$M_{ADM} \geq \frac{1}{2}\left(\frac{|\partial\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \qquad \omega_{n-1} \text{ area of the standard unit sphere}$$

and equality occurs if and only if (Σ, g) is the Schwarzschild n-space with $m = M_{ADM}$.

Interesting problem in Riemannian geometry.

Riemannian Penrose inequality conjecture in low dimensions

- Major breakthroughs for the Penrose inequality at the turn of the past century.
- First breakthrough was due to Huisken and Ilmanen (Riemannian Penrose inequality in dimension 3 for connected boundary).

Theorem (Huisken & Ilmanen '97)

Let (Σ, g) be 3-dimensional, asymptotically flat with outermost minimal boundary $\partial\Sigma$ and satisfying $R_g \geq 0$. Let $\{\partial_a\Sigma\}$ be the connected components of $\partial\Sigma$. Then

$$M_{ADM}(g) \geq \max_a \sqrt{\frac{|\partial_a\Sigma|}{16\pi}}.$$

Moreover, equality occurs if and only of (Σ, g) is the Schwarzschild 3-space.

- Second breakthrough by Bray. No connectedness assumption.

Theorem (Bray '99)

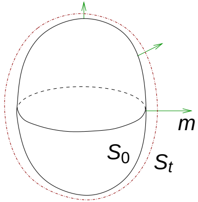
The Riemannian Penrose inequality conjecture in dimension three holds true.

Strategy in Huisken & Ilmanen's proof

Their starting point is an early heuristic argument [Geroch '73], [Jang & Wald '77] :

- The Hawking mass $M_G(S) = \sqrt{\frac{|S|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S H_S^2 dS\right)$ is monotonically increasing under inverse mean curvature flow provided S is connected.

Flow of surfaces:



- Smooth map $F : S \times I \to (\Sigma, g)$, $I \subset \mathbb{R}$ interval, such that $\phi_t := F(\cdot, t)$ is embedding.
- m_t a unit normal to $S_t := \phi_t(S)$ and H_t its mean curvature.
- $\{S_t\}$ define an inverse mean curvature flow iff $F_\star(\cdot, \partial_t) = \frac{1}{H_t} m_t$.

- The Hawking mass approaches the ADM mass for large coordinate spheres in the asymptotically flat end of (Σ, g).

The Riemannian Penrose inequality follows if there is an inverse mean curvature flow by connected surfaces interpolating between $\partial\Sigma$ and large coordinates.

- Huisken & Ilmanen: Such smooth flow need not exist, but a suitable weak formulation can be made to work.
- Key analytic tool: Use the level sets of solutions of the PDE

$$\mathrm{div}_g\left(\frac{\mathrm{grad}_g u}{|\mathrm{grad}_g u|}\right) = |\mathrm{grad}_g u|, \qquad u = 0 \quad \text{on} \quad \partial\Sigma.$$

Strategy in Bray's proof

- Key idea: use a flow of metrics interpolating between g and Schwarzschild metric.

A conformal flow is a family of metrics $\{g_t\}$, $t \in \mathbb{R}^+$ on Σ satisfying:

- $g_t(x)$ is C^1 in x, Lipschitz in t and satisfies $\frac{dg_t}{dt} = 4v_t g_t$ with v_t solving:

$$\left.\begin{array}{ll} \Delta_{g_t} v_t = 0 & \text{on } \Omega^+(S_t) \\ v_t = 0 & \text{on } \Sigma \setminus \Omega^+(S_t) \\ \lim_{x\to\infty} v_t = -1 & \end{array}\right\}$$

where S_t is the minimal area enclosure of $\partial\Sigma$ in (Σ, g_t).

Structure of Bray's proof:

With (Σ, g) as in the Riemannian Penrose inequality a conformal flow exists with $g_0 = g$ and satisfies:

- $|S_t|_{g_t} = |\partial\Sigma|_g$ for all $t \geq 0$.
- $\lim_{t\to\infty} \Omega^+(S_t) = \emptyset$
- The ADM mass $M_{ADM}(t)$ of (Σ, g_t) is monotonically decreasing.
- After a t-dependent diffeomorphism $(\Omega^+(S_t), g_t)$ converges in a suitable sense to a Schwarzschild space of mass $m \geq \sqrt{|\partial\Sigma|_g/(16\pi)}$.

These properties imply the Riemannian Penrose conjecture:

$$M_{ADM}(g) = M_{ADM}(0) \geq M_{ADM}(t) \geq \lim_{t\to\infty} M_{ADM}(t) \geq m \geq \sqrt{\frac{|\partial\Sigma|_g}{16\pi}}.$$

- The Huisken-Ilmanen argument is inherently 3-dimensional, Bray's method can be extended to higher dimensions.

Theorem (Bray & Lee '09)

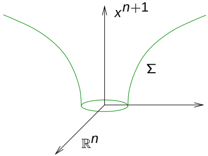
Let (Σ^n, g) be asymptotically flat, with compact outermost minimal boundary $\partial\Sigma$ and satisfying $R_g \geq 0$. If $3 \leq n \leq 7$ then

$$M_{ADM}(g) \geq \frac{1}{2}\left(\frac{|\partial\Sigma|}{\omega_{n-1}}\right)^{\frac{n-1}{n-2}}.$$

The rigidity part in dimensions $4 \leq n \leq 7$ is proven under a topological restriction on Σ (requires the manifold to be spin).

Riemannian Penrose inequality for graphs in Euclidean space

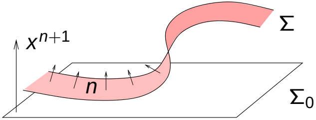
- The Schwarzschild n-space of mass $m > 0$ can be isometrically embedded as a graph in Euclidean space.
- The inner boundary is a sphere of radius $(2m)^{1/(n-2)}$ embedded in a hyperplane (taken e.g. as $x^{n+1} = 0$)



Natural to ask whether the Riemannian Penrose inequality holds for appropriate graphs in Euclidean space [Lam '10].

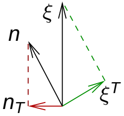
General setup:

- (Σ_0, h) n-dim Riemannian manifold, $\mathrm{I} \subset \mathbb{R}$ interval.
- $N = \Sigma_0 \times \mathrm{I}$ with metric $\gamma = h + (dx^{n+1})^2$.
- Σ orientable embedded hypersurface in (N, γ), unit normal n, induced metric g, K second fundamental form.



Define ξ vector field tangent to the I factor with $\xi(x^{n+1}) = 1$. Killing field $\mathcal{L}_\xi \gamma = 0$.

- Decompose



Define

- $\xi = N\, n + \xi^T$
- $n = \alpha\, \xi + n_T$

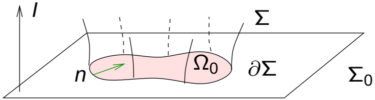
A general identity follows [Lam '10, Lopes de Lima & Girao '12]:

$$\mathrm{div}_g\left((K-kg)(\xi^T)\right) = N\left(R_g - R_h + Ric_h(n_T, n_T)\right).$$

Strategy: integrate this identity on Σ and use the Gauss theorem.

- Assume for simplicity $\{\partial\Sigma\}$ connected (not necessary). Enforce minimal boundary as follows:

- $\partial\Sigma$ embedded in $\Sigma_0 := \{x^{n+1} = 0\}$and $n|_{\partial\Sigma}$ tangential to Σ_0.
- $\partial\Sigma = \partial\Omega$ with Ω a domain in Σ_0.



Assume (Σ_0, h) is asymptotically flat with mass $M_{ADM}(h)$ and $\Sigma \setminus \partial\Sigma$ is an asymptotically flat graph over $\Sigma_0 \setminus \overline{\Omega}$.

- The boundary integral of $(K-kg)(\xi^T)$ "at infinity" gives $M_{ADM}(g) - M_{ADM}(h)$.
- At the inner boundary gives the mean curvature H of $\partial\Omega \hookrightarrow (\Sigma_0, h)$.

Proposition ([Lam '10])

$$M_{ADM}(g) = M_{ADM}(h) + c_n\left(\int_{\partial\Omega} H d(\partial\Omega) + \int_{\Sigma} N\left(R_g - R_h + Ric_h(n_T, n_T)\right) d\Sigma\right).$$

Recall $c_n = \frac{1}{2(n-1)\omega_{n-1}}$.

Leads to the Riemannian Penrose inequality for graphs.

Theorem ([Lam '10])

Let (Σ, g) be an asymptotically flat graph over the hyperplane $\{x^{n+1} = 0\}$ in $(\mathbb{R}^{n+1}, \delta)$ $(n \geq 3)$. Assume that $\partial\Sigma$ is compact and the boundary of a mean convex, star-shaped domain $\Omega \subset \{x^{n+1} = 0\}$. If $R_g \geq 0$ and Σ is orthogonal to $\{x^{n+1} = 0\}$ along $\partial\Sigma$, then

$$M_{ADM}(g) \geq \frac{1}{2}\left(\frac{|\partial\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

The classic Minkowski inequality states $c_n \int_{\partial\Omega} H d(\partial\Omega) \geq \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$ for any convex domain Ω with smooth and compact boundary in Euclidean n-space.

- Generalized to star-shaped domain with mean convex boundary by [Guan & Li '09].

We are in the general setup above with $(\Sigma_0, h) = (\mathbb{R}^n, \delta)$.

- Graph condition imposes $N > 0$ on $\mathrm{int}(\Sigma)$. So, $NR_g \geq 0$ and

$$M_{ADM}(g) \geq c_n \int_{\partial\Omega} H d(\partial\Omega) \geq \frac{1}{2}\left(\frac{|\partial\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

- Except for spherical symmetry, this is the only case where the Riemannian Penrose inequality is known to hold in arbitrary dimension.

- Equality in the Minkowski inequality occurs iff $\partial\Omega$ is a round sphere.
- Equality in the Riemannian Penrose inequality for graphs implies $R_g = 0$ and $\partial\Sigma$ is a round sphere.
- $R_g = 0$ is a fully non-linear equation for the graph function $\{x^{n+1} = f\}$, $f \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega}, \mathbb{R})$.
- Proving uniqueness is a non-trivial problem.

Theorem (Huang & Wu '12)

Let Σ be as in the previous theorem. If

$$M_{ADM}(g) = \frac{1}{2}\left(\frac{|\partial\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

then (Σ, g) is isometric to the Schwarzschild n-space.

The Bray and Khuri approach to the general Penrose inequality

- The Penrose inequality conjecture in the non-time symmetric case is a hard problem.
- (Partially) proven only in dimension 3 and spherical symmetry.
- Interesting proposal by [Bray & Khuri '09] to address the general case.

It is natural to ask whether the general case can be reduced to the Riemannian case.

- Deform the metric g in (Σ, g, K) to another metric $\overline{g}$ with $R_{\overline{g}} \geq 0$ and apply the Riemannian Penrose inequality.

Successful approach for the Positive Energy Theorem in the non-time symmetric case [Schoen & Yau '79]:

- Jang deformation [Jang '78]: $\overline{g} = g + df \otimes df$, $f : \Sigma \to \mathbb{R}$ solving the Jang equation.

$$\mathrm{tr}_{\overline{g}} \left(\frac{\mathrm{Hess}_g f}{\sqrt{1 + |df|_g^2}} - K \right) = .0$$

- Unsuitable approach for the general Penrose inequality [Malec & O'Murchadha 2004].

Bray & Khuri propose a modified deformation and set of equations:

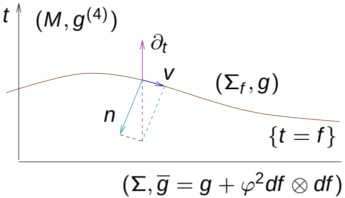
- Modified Jang transformation:

$$\overline{g} = g + \varphi^2 df \otimes df, \quad f \in C^\infty(\Sigma, \mathbb{R}), \quad \varphi \in C^\infty(\Sigma, \mathbb{R}^+).$$

Given f, φ define an auxiliary spacetime $M = \Sigma \times \mathbb{R}$, $g^{(4)} = -\varphi^2 dt^2 + (g + \varphi^2 df \otimes df)$.

Define $\Sigma_f := \{t = f\}$:

- h: Second fundamental form w.r.t past directed unit normal n.
- v: tangent vector to Σ_f such that $\partial_t = N(n + v)$



Proposition (Bray & Khuri, 2009)

If f and φ satisfy the **generalized Jang equation**, $tr_{\overline{g}}(h - K) = 0$, *then,*
$R_{\overline{g}} = 16\pi(\rho - \boldsymbol{J}(v)) + |h - K|^2_{\overline{g}} + 2|q|^2_{\overline{g}} + \varphi^{-1} div_{\overline{g}}\left(\varphi(h - K)(\vec{v}, \cdot)\right)$.

2 unknowns $\longrightarrow$ Need for a second PDE. Bray and Khuri make two proposals:

- **Divergence equation:** $div_{\overline{g}}\left(\varphi(h - K)(\vec{v}, \cdot)\right) = 0$ (makes $R(\overline{g}) \geq 0$).
- **Jang-IMCF equation:** $\varphi = |\overline{D}u|_{\overline{g}} e^{u/2}$, u solution of weak IMCF in $(\Sigma, \overline{g})$ well-suited for applying the Huisken-Ilmanen method on $(\Sigma, \overline{g})$

Both imply sufficient positivity of $R(\overline{g})$ to apply the Riemannian Penrose inequality.

Main issue: Existence of solutions under appropriate boundary conditions

- Existence of solutions of the generalized Jang equation for prescribed φ [Han & Khuri '12].

Final remarks

The Penrose inequality conjecture is an important problem in General Relativity and in Geometric Analysis.

- Proving the conjecture would give indirect support to the weak cosmic censorship conjecture and would strengthen the positive mass theorem.
- Finding counterexamples would indicate that the weak cosmic censorship conjecture might be false.

In this talk I have left out many issues concerning the Penrose inequality conjecture:

- There are versions of the Penrose inequality conjecture involving asymptotically hyperbolic initial data sets or null hypersurfaces approaching null infinity.
- There exist a number of partial, or suboptimal, results concerning this inequality in several situations.
- There exist stronger versions involving the total charge of the spacetime (and/or the total angular momentum under additional symmetry assumptions) and interesting recent results on these.
- The Penrose conjecture can be used to derive conjectures involving the geometry of surfaces in simple spacetimes like Minkowski or Schwarzschild.
- Etc.