Fig 1

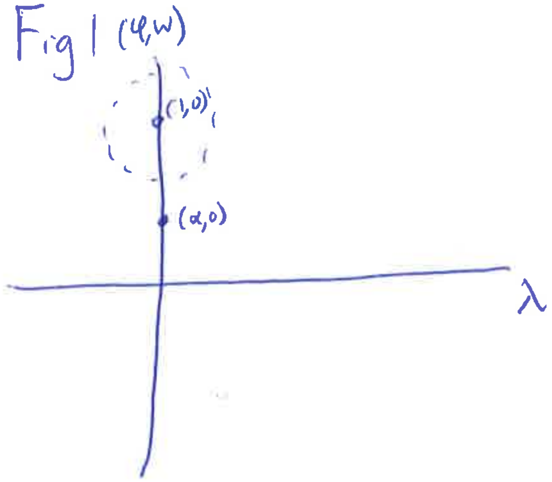
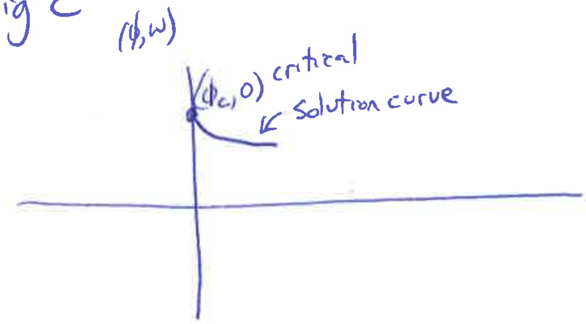


Fig 2



APPLICATIONS OF BIFURCATION THEORY TO THE EINSTEIN CONSTRAINT EQUATIONS

CALEB MEIER

Slide 25:

$$D_xF((c,0),0) = \begin{bmatrix} -\Delta & 0 \\ 0 & \mathbb{L} \end{bmatrix}$$

where $\mathbb{L} = \text{div}\mathcal{L}$.

Slide 30: The problem with the proof is that if you have some graph of the solution space (see figure 1), we know we have solutions for small λ (of the equations on slide 24), but the solutions may not be in the neighborhood of $(1,0)$. However, it is conceivable they live in a neighborhood of some $(\alpha, 0)$ for some α sufficiently small. This came to my attention because the statement of the theorem as I gave contradicts the case when $\tau \equiv 0$ and $\sigma \not\equiv 0$. In that case, solutions exist for positive scalar curvature metrics, but do *not* exist for Yamabe zero or negative metrics.

Slide 32: See figure 2. I want my solution curve to be more exciting than just constant, sitting at the critical solution $(\phi_c, 0)$. I want it to either cross (into negative Yamabe metrics) or double back (to get non-uniqueness). This is why I need analyticity.

Applications of Bifurcation Theory to the Einstein Constraint Equations

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Outline

The Einstein Constraint Equations

Using the $3+1$ decomposition of spacetime, one can formulate the Einstein Equations as an initial value problem where the initial data consists of a Riemannian metric $\hat{g}_{ab}$ and a symmetric tensor $\hat{k}_{ab}$ on a specified 3-dimensional manifold $\mathcal{M}$.

Like Maxwell's equations, the initial data $\hat{g}_{ab}$ and $\hat{k}_{ab}$ must satisfy constraint equations, where the constraints take the form

Definition 1

$$\hat{R} + \hat{k}^{ab}\hat{k}_{ab} + \hat{k}^2 = 2\kappa\hat{\rho}, \tag{1}$$

$$\hat{D}_b\hat{k}^{ab} - \hat{D}^a\hat{k} = \kappa\hat{j}^a. \tag{2}$$

Here $\hat{R}$ and $\hat{D}$ are the scalar curvature and covariant derivative associated with $\hat{g}_{ab}$, $\hat{k}$ is the trace of $\hat{k}_{ab}$ and $\hat{\rho}$ and $\hat{j}^a$ are matter terms obtained by contracting the stress energy tensor with a vector field normal to $\mathcal{M}$.

The Conformal Method

The *York conformal decomposition* splits initial data into freely specifiable pieces plus 4 pieces determined by the constraints. First conformally transform the metric $\hat{g}_{ab}$ to obtain:

- $\hat{g}_{ab} = \phi^4 g_{ab}$,
- $\hat{\tau} = \hat{k}_{ab}\hat{g}^{ab} = \tau$.

Then decompose $\hat{k}_{ab}$ into its trace and its symmetric, trace-free part $\hat{l}^{ab}$:

- $\hat{k}^{ab} - \frac{1}{3}\hat{g}^{ab}\hat{\tau} = \hat{l}^{ab}$

Then rescale the symmetric, trace free tensor to obtain a new symmetric, tracefree tensor l^{ab}, where

- $\hat{l}^{ab} = \phi^{-10} l^{ab}$.

Using a general algebraic result, we may decompose l^{ab} in the following way:

- $l^{ab} = \sigma^{ab} + \mathcal{L}w^{ab}, \quad \mathcal{L}w^{ab} = D^a w^b + D^b w^a - \frac{2}{3} g^{ab} D_k w^k,$

where σ^{ab} is symmetric, traceless and divergence free ($D_b\sigma^{ab} = 0$).

Unscaled CTT Formulation

Making the above substitutions for $\hat{k}_{ab}$ into the constraint equations

$$^{(3)}\hat{R} + \hat{\tau}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \qquad \hat{D}^a\hat{\tau} - \hat{D}_b\hat{k}^{ab} - \kappa\hat{j}^a = 0,$$

and using the fact that $^3\hat{R} = \phi^{-5}(^3R\phi - 8\Delta\phi)$, we obtain a coupled elliptic system for the conformal factor ϕ and w^a:

Definition 2 (Unscaled CTT Equations)

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})(\sigma^{ab} + (\mathcal{L}w)^{ab})\phi^{-7} - 2\kappa\hat{\rho}\phi^5 = 0,$$

$$-D_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6 D^b\tau + \kappa\phi^{10}\hat{j}^b = 0.$$

Scaled CTT Formulation

Making the same substitutions as in the unscaled case and letting

$$\hat{\rho} = \phi^{-8}\rho \quad \text{and} \quad \hat{j}^a = \phi^{-10} j^a,$$

we obtain the more standard scaled CTT formulation of the constraints:

Definition 3 (Scaled CTT Equations)

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})(\sigma^{ab} + (\mathcal{L}w)^{ab})\phi^{-7} - 2\kappa\rho\phi^{-3} = 0,$$

$$-D_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6 D^b\tau + \kappa j^b = 0.$$

Free data and Determined Data

Both (2) and (3) are determined systems of elliptic PDE with free data:

- g_{ab} - conformally related metric,
- σ_{ab} - symmetric, traceless, divergence free tensor,
- τ- the mean curvature function,
- $\hat{\rho}, \hat{j}^a$- energy density and momentum current density

and determined data:

- ϕ - conformal factor (unknown portion of metric)
- **w** - unknown portion of extrinsic curvature k_{ab}.

By solving the unscaled CTT formulation for (ϕ, w), one obtains the following physical solutions to the Einstein constraints:

- $\hat{g}_{ab} = \phi^4 g_{ab}$
- $\hat{k}^{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \frac{1}{3}\phi^{-4}\tau g^{ab}$.

By determining solutions these equations one is parametrizing solutions to the constraints by the freely specifiable data.

Non-uniqueness of Unscaled Equations

Equations of the form of the Unscaled CTT equations arise in the study of the Einstein-field equation [4] and in the conformal thin sandwich formulation of the constraints with unscaled sources [15, 2]. In [14] we consider the uniqueness properties of solutions to equations of this form on closed manifolds. The motivation for this work stemmed from the following:

- The semilinear portion of unscaled CTT equations is not necessarily monotone and Hamiltonian constraint can have non-convex energy, so uniqueness is not expected.
- A partial of analysis of the non-uniqueness properties of equations similar to (2) is given in [2, 15] in the asymptotically Euclidean setting.

Solution Theory for Scaled CTT Equations

Solution Theory for coupled, scaled CTT equations on closed $\mathcal{M}$ depends greatly on (g, τ, σ).

- τ is constant, CTT equations decouple. Solution theory is well-understood for all Yamabe classes and σ in low regularity setting. Solutions are unique when they exist. [Choquet-Bruhat, Isenberg, Maxwell, Holst, Nagy, Tsogtgerel][3, 8, 10, 6]
- τ is near-constant (near-CMC $\frac{\|d\tau\|_z}{\min \tau} \leq C$). Solution theory is well-understood for all Yamabe classes and σ in low regularity setting. Solutions are unique when they exist. [Isenberg, Moncrief, Allen, Clausen, Holst, Nagy, Tsogtgerel] [9, 1, 6]
- τ is far-from-CMC (no restriction on τ). Solutions exist for low regularity data in the event that $g \in \mathcal{Y}^+$ and σ is sufficiently small. Solutions not necessarily unique. [Holst, Nagy, Tsogtgerel, Maxwell][6, 7, 11]

Non-uniqueness of Scaled Equations

In [13], we consider the uniqueness properties of the solutions obtained in [6] to the scaled equations. The primary motivation for this work stemmed from the following:

- The far-from-CMC existence results in [6] rely on the Schauder fixed point Theorem, which do not guarantee uniqueness of solutions.
- In [12], Maxwell showed that solutions to the scaled CTT equations are non-unique for metrics in the zero Yamabe class and families of low-regularity mean curvature functions ($\tau \in L^\infty$ as opposed to $\tau \in W^{1,z}$ in [6]).

General Approach

The approach to analyzing the uniqueness properties of both the scaled and unscaled CTT equations in [14] and [13] is the same and is outlined in the following steps.

1. Fix some one parameter family of data $(g_\lambda, \tau_\lambda, \sigma_\lambda, \rho_\lambda, j^a_\lambda)$, where $\lambda \in \mathbb{R}$.
2. Formulate the CTT equations as a nonlinear problem between Banach spaces where solutions $(\phi, \mathbf{w})$ satisfy $F((\phi, \mathbf{w}), \lambda) = 0$ for some λ.
3. Find solutions $((\phi_0, \mathbf{w}_0), \lambda_0)$ where the linearization of $F((\phi, \mathbf{w}), \lambda)$ has a one-dimensional kernel.
4. Apply a Liapunov-Schmidt reduction to parametrize solution curve in a neighborhood of $((\phi_0, \mathbf{w}_0), \lambda_0)$
5. Analyze solution curve using a Taylor series expansion or some other means.

Liapunov-Schmidt Reduction

Let X, Λ and Z be Banach spaces and $U \subset X$, $V \subset \Lambda$. Suppose $F : U \times V \to Z$ is a nonlinear Fredholm operator of index zero with respect to x that also satisfies

$$F(x_0, \lambda_0) = 0 \quad \text{for some } (x_0, \lambda_0) \in U \times V,$$
$$\dim \ker(D_x F(x_0, \lambda_0)) = \dim \ker(D_x F(x_0, \lambda_0)^*) = 1.$$

X and Z decompose with respect to $D_x F(x_0, \lambda_0)$ and define projection operators P and Q satisfying

$$P : X \to X_1 = \ker(D_x F(x_0, \lambda_0)), \quad Q : Y \to Y_2 = \ker(D_x F(x_0, \lambda_0)^*).$$

Then $F(x, \lambda) = 0$ if and only if the following two equations are satisfied

$$QF(x, \lambda) = 0, \qquad (3)$$
$$(I - Q)F(x, \lambda) = 0.$$

Write $x = v + w$, $(x_0 = v_0 + w_0)$, where $v = Px$ and $w = (I - P)x$. Apply Implicit Function Theorem to the operator

$$G(v, w, \lambda) = (I - Q)F(v + w, \lambda),$$

to conclude that there exists

$$\begin{array}{ll} \psi : U \times V \to W & \text{such that all solutions to } G(v, w, \lambda) = 0 \\ \text{in } U \times W \times V & \text{are of the form } G(v, \psi(v, \lambda), \lambda) = 0. \end{array}$$

Insertion of the function $\psi(v, \lambda)$ into (3) yields a finite-dimensional problem

$$\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) = 0. \tag{4}$$

With added conditions on $F(x, \lambda)$, one can apply the Implicit Function Theorem to $\Phi(v, \lambda)$ to conclude that there exists

$$\gamma : U_1 \to V_1, \quad \gamma(v_0) = \lambda_0, \quad \Phi(v, \gamma(v)) = 0.$$

Parametrization of Solution Curve

Inserting $\gamma(v)$ into Eq. (4) we obtain

$$g(v) = QF(v + \psi(v, \gamma(v)), \gamma(v)). \tag{5}$$

By writing $v = s\hat{v}_0 + v_0$ and inserting this into (5), we obtain the solution curve

$$x(s) = v_0 + s\hat{v}_0 + \psi(v_0 + s\hat{v}_0, \gamma(v_0 + s\hat{v}_0)), \tag{6}$$

$$\lambda(s) = \gamma(v_0 + s\hat{v}_0). \tag{7}$$

With added assumptions on $F(x, \lambda)$, we may determine $\ddot{\lambda}(0)$ to determine second order Taylor expansions of

$$\lambda(s) \quad \text{and} \quad f(s) = \psi(v_0 + s\hat{v}_0, \gamma(v_0 + s\hat{v}_0)).$$

Problem Considered

We consider the following one-parameter family of problems

$$-\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\hat{\rho}e^{-\lambda}\phi^5 = 0, \quad (8)$$
$$\mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 = 0$$

on a closed manifold $(\mathcal{M}, g_{ab})$.

We observe that Eq.(8) is a family of unscaled CTT equations with specified data $(g, \lambda\tau, \sigma, e^{-\lambda}\hat{\rho}, \mathbf{0})$, where

$$a_R = \frac{1}{8}R, \qquad a_\tau = \frac{1}{12}\tau^2, \quad (9)$$
$$a_{\mathbf{w}} = \frac{1}{8}[\sigma_{ab} + (\mathcal{L}\mathbf{w})_{ab}][\sigma^{ab} + (\mathcal{L}\mathbf{w})^{ab}], \qquad b_\tau = \frac{2}{3}D^a\tau.$$

Set-Up of Problem

We reformulate Eq. (8) so that we can apply techniques from functional analysis and bifurcation theory to analyze the problem. Define

$$F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\hat{\rho}e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 \end{bmatrix}. \qquad (10)$$

We view (10) as a nonlinear operator between Banach spaces

$$F((\phi,\mathbf{w}),\lambda) : C^{k,\alpha}(\mathcal{M}) \oplus C^{k,\alpha}(\mathcal{TM}) \times \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}) \oplus C^{k-2,\alpha}(\mathcal{TM}),$$

where ($k \geq 2$). If $F((\phi,\mathbf{w}),\lambda) = 0$, then $((\phi,\mathbf{w}),\lambda)$ solves Eq. (8).

Criterion for non-uniqueness

- Want to show that solutions to $F((\phi, \mathbf{w}), \lambda) = 0$ are non-unique in neighborhood of some point $((\phi_0, \mathbf{w}_0), \lambda_0)$.
- If $X = (\phi, \mathbf{w})$, Implicit Function Theorem says that if $D_X F((\phi_0, \mathbf{w}_0), \lambda_0)$ is invertible, then solution can be uniquely parametrized by λ in a neighborhood of $((\phi_0, \mathbf{w}_0), \lambda_0)$.
- In order for solutions to be non-unique, we must find a point where $D_X F((\phi_0, \mathbf{w}_0), \lambda_0)$ is not invertible. (Note that this condition is not sufficient for non-uniqueness, only necessary.)

Main Results: Existence of Critical Density

Theorem 4

Suppose that R and $|\sigma|$ are constant. Let $D_X F((\phi, \mathbf{w}), \lambda)$ denote the Fréchet derivative of (9) with respect to $X = (\phi, \mathbf{w})$ and let ρ_c and ϕ_c be defined by

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}\pi|\sigma|} \quad \text{and} \quad \phi_c = \left(\frac{R}{24\pi\rho}\right)^{\frac{1}{4}}. \tag{11}$$

Then when $\rho = \rho_c$, $\dim \ker(D_X F((\phi_c, \mathbf{0}), 0)) = 1$ and it is spanned by the constant vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Theorem 5

Suppose that $\tau \in C^{1,\alpha}(\mathcal{M})$ *and let* $F((\phi, \mathbf{w}), \lambda)$ *be defined as in* (10). *Then if* ρ_c *and* ϕ_c *are defined as in Theorem* 4 *and* $\rho = \rho_c$, *there exists a neighborhood of* $((\phi_c, \mathbf{0}), 0)$ *such that all solutions to* $F((\phi, \mathbf{w}), \lambda) = 0$ *in this neighborhood lie on a smooth curve of the form*

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{12}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3),$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3),$$

where $u(x) \in C^{2,\alpha}(\mathcal{M})$ *and* $\mathbf{0} \neq \mathbf{w}(x) \in C^{2,\alpha}(\mathcal{TM})$. *In particular, there exists a* $\delta > 0$ *such that for all* $0 < \lambda < \delta$ *there exist elements* $(\phi_{1,\lambda}, \mathbf{w}_{1,\lambda}), (\phi_{2,\lambda}, \mathbf{w}_{2,\lambda}) \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ *such that*

$$F((\phi_{i,\lambda}, \mathbf{w}_{i,\lambda}), \lambda) = 0, \quad \text{for } i \in \{1,2\}, \quad \text{and } (\phi_{1,\lambda}, \mathbf{w}_{1,\lambda}) \neq (\phi_{2,\lambda}, \mathbf{w}_{2,\lambda}).$$

Outline of Proof for Critical Density

Define the polynomial

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

where ρ_c is to be determined. Then we do the following:

- Require that $q(\chi)$ have a single, positive double root. This condition determines $\phi_c,\ \rho_c$.
- Apply the maximum principle to the problem

$$\Delta\phi = a_R\phi - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho\phi^5, \tag{13}$$

  when $\rho > \rho_c$ to conclude no solutions exist.
- Apply method of sub-and super-solutions to Eq. (13) when $\rho \le \rho_c$ to conclude solutions exist.
- Compute $D_X F((\phi_c, \mathbf{0}), 0)$ and use properties of $q(\chi)$ to conclude that it has a one-dimensional kernel

Non-uniqueness Proof

- Show that the operator $D_XF((\phi_c,\mathbf{0}),0)$ is a Fredholm operator between $C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})$.
- Use the fact that $\dim\ker(D_XF((\phi_c,\mathbf{0}),0))=1$ to decompose the domain and codomain

$$C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})=$$

$$\ker(D_XF((\phi_c,\mathbf{0}),0))\oplus(R(D_XF((\phi_c,\mathbf{0}),0)^*)\cap(C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM}))),$$

$$C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})=$$

$$(R(D_XF((\phi_c,\mathbf{0}),0))\cap(C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})))\oplus\ker(D_XF((\phi_c,\mathbf{0}),0)^*).$$

Non-uniqueness Proof

- Apply the Liapunov-Schmidt Reduction and use the fact that $D_\lambda F((\phi_c, \mathbf{0}, 0) \neq 0$ to parametrize solution curve in a neighborhood of $(\phi_c, \mathbf{0}, 0)$

$$\begin{bmatrix} \phi(s) \\ \mathbf{w}(s) \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix} + \psi\left(s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}, \gamma\left(s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}\right)\right),$$

$$\lambda(s) = \gamma\left(s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}\right),$$

- Use fact that $D^2_{XX}F((\phi_c, \mathbf{0}, 0)[\hat{v}_0, \hat{v}_0] \notin R(D_X F((\phi_c, \mathbf{0}, 0))$ to conclude that $\ddot{\lambda}(0) \neq 0$ to determine second order Taylor series of $\lambda(s)$, $\mathbf{w}(s)$ and $f(s) = \psi(\phi_c \hat{v}_0 + s\hat{v}_0, \gamma(\phi_c \hat{v}_0 + s\hat{v}_0))$ to get

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{14}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3),$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3),$$

Analysis of Solution Curve

Asymptotic properties of solution curve $((\phi(s), \mathbf{w}(s)), \lambda(s))$ imply the following:

- For s small, there will exist s_1 and s_2 such that $\lambda(s_1) = \lambda(s_2)$
- For s small, $\phi(s)$ is one-to-one

These two properties imply that a **saddle node bifurcation** (or fold) occurs at $((\phi_c, \mathbf{0}), 0)$, and that there exists s_1, s_2 such that $\lambda(s_1) = \lambda(s_2) = \lambda_0$ and $((\phi(s_1), \mathbf{w}(s_1), \lambda_0) \neq ((\phi(s_2), \mathbf{w}(s_2), \lambda_0)$.

This implies that both $((\phi(s_1), \mathbf{w}(s_1))$ and $((\phi(s_2), \mathbf{w}(s_2))$ satisfy the unscaled CTT equations with specified data $(g, \lambda_0\tau, \sigma, e^{-\lambda_0}\hat{\rho}, \hat{j}^a)$.

Problem Considered

In case of Scaled CTT equations, we consider the one-parameter family of nonlinear problems of problems

$$
\begin{aligned}
-\Delta_\lambda\phi+\frac{1}{8}\lambda\phi+\frac{\lambda^4}{12}\tau^2\phi^5-\frac{1}{8}(\lambda^2\sigma+\mathcal{L}\mathbf{w})_{ab}(\lambda^2\sigma+\mathcal{L}\mathbf{w})^{ab}\phi^{-7}-\frac{\lambda^2\kappa}{4}\rho\phi^{-3}&=0,\\
\mathbb{L}_\lambda\mathbf{w}+\frac{2\lambda^2}{3}D_\lambda\tau\phi^6+\lambda^2\kappa j^a&=0.
\end{aligned} \tag{15}
$$

Eq. (15) represents the scaled CTT equations with a one-parameter family of data $(g_\lambda,\tau_\lambda,\sigma_\lambda,\rho_\lambda,j^a_\lambda)$, where g_λ is a one parameter family of metrics satisfying $R(g_\lambda)=\lambda$ and

$$\tau_\lambda=\lambda^2\tau,\ \ \sigma_\lambda=\lambda^2\sigma,\ \ \rho_\lambda=\lambda^2\rho,\ \ \text{and}\ \ j^a_\lambda=\lambda^2 j^a.$$

Set-Up

As in unscaled CTT case, we let

$$F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta_\lambda\phi + \lambda a_R\phi + \lambda^4 a_\tau\phi^5 - a_{\mathbf{w},\lambda}\phi^{-7} - \lambda^2 a_\rho\phi^{-3} \\ \mathbb{L}_\lambda \mathbf{w} + \lambda^2 b_\tau\phi^6 + \lambda^2 b_{\mathbf{j}} \end{bmatrix},$$

where

$$a_R = \frac{1}{8}, \quad a_{\mathbf{w},\lambda} = \frac{1}{8}(\lambda^2\sigma + \mathcal{L}\mathbf{w})_{ab}(\lambda^2\sigma + \mathcal{L}\mathbf{w})^{ab},$$
$$a_\tau = \frac{1}{12}\tau^2, \quad b_\tau = \frac{2}{3}D_\lambda\tau, \quad a_\rho = \frac{\kappa}{4}\rho, \quad b_{\mathbf{j}} = \kappa\mathbf{j}.$$

Solutions to scaled CTT equations satsify $F((\phi,\mathbf{w}),\lambda) = 0$ and

$$F((\phi,\mathbf{w}),\lambda) : C^{k,\alpha}(\mathcal{M}) \oplus C^{k,\alpha}(\mathcal{TM}) \times \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}) \oplus C^{k-2,\alpha}(\mathcal{TM}).$$

Using an Implicit Function Theorem argument, we have the following:

Theorem 6

Suppose that $\mathcal{M}$ is a closed 3-dimensional manifold that admits a metric with positive scalar curvature. Then for $\lambda \in U$, where U is a neighborhood of 0, there exists a one-parameter family of metrics (g_λ) through g_0 such that $R(g_\lambda) = \lambda$. Moreover, $g_\lambda : U \to \mathcal{A}^{s,p}$ is analytic.

Theorem 6 shows that the one-parameter family of operators $F((\phi, \mathbf{w}), \lambda)$ is meaningful, and by applying a Liapunov-Schmidt reduction and analyzing the solution curve of $F((\phi, \mathbf{w}), \lambda) = 0$ in a neighborhood of $((1, \mathbf{0}), 0)$, we have the following Theorem.

Theorem 7

Let $\mathcal{M}$ be a closed 3-dimensional manifold which admits both a metric with positive scalar curvature and a metric g_0 with zero scalar curvature and no conformal Killing fields, where both metrics are contained in $\mathcal{A}^{s,p}$, $s > 3 + \frac{3}{p}$. Let $(\tau, \sigma, \rho, \mathbf{j})$ be freely specified data for the CTT formulation of the constraints. Then in any neighborhood U of g_0 there exists a metric $g \in W^{s,p}$ and a $\lambda > 0$ such that at least one the following must hold:

- *$R(g) = \lambda$ and solutions to the CTT formulation of the Einstein Constraints with specified data $(g, \lambda^2\tau, \lambda^2\sigma, \lambda^2\rho, \lambda^2\mathbf{j})$ are non-unique*
- *$R(g) = -\lambda$ and there exists a solution to CTT formulation of the Einstein Constraints with specified data $(g, \lambda^2\tau, \lambda^2\sigma, \lambda^2\rho, \lambda^2\mathbf{j})$.*

Thus, in any neighborhood of a metric with zero scalar curvature and no conformal Killing fields, either there exists a Yamabe positive metric for which solutions to the CTT formulation are non-unique or there exists a Yamabe negative metric for which far-from-CMC solutions to the CTT formulation exist.

Analyticity of $R(g)$

First show that the operator $R : \mathcal{A}^{s,p} \to W^{s-2,p}$ is an analytic operator.

- We have that $D^k R(g) h^k = 0$ for $k \geq 8$, where $D^k R(g)$ is the k-th Frechet derivative of R at g and

$$h^k = \underbrace{(h, \cdots, h)}_{k \text{ times}}.$$

- This implies that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} D^k R(g_0) h^k$$

converges absolutely for arbitrary $h \in \mathcal{A}^{s,p}$.

- Taylor's Theorem then implies implies the analyticity of R.

Outline of Proof for Family of Metrics

Construct one parameter family using an Implicit Function Theorem argument

- Letting $\mathcal{S}^{s,p}$ denote all symmetric two tensors in $W^{s,p}$, the splitting result in [5] implies that

$$\mathcal{S}^{s,p} = \ker(DR(g_0)) \oplus \operatorname{Ran}(DR(g_0)^*)$$

  as long as g_0 is non-flat.
- Using this result, for $h \in \operatorname{Ran}(DR(g_0)^*)$ small, $g_0 + h$ defines a neighborhood of g_0 in $\mathcal{S}^{s,p}$ and $G(h, \lambda) = R(g_0 + h) - \lambda$ is well-defined. Apply Implicit Function Theorem to $G(h, \lambda)$ to obtain $\psi(\lambda)$ such that $0 = G(\psi(\lambda), \lambda) = R(g_0 + \psi(\lambda)) - \lambda$.
- $g_\lambda = g_0 + \psi(\lambda)$ has same regularity as $G(h, \lambda)$.

By design, linearization of $F((\phi, \mathbf{w}), \lambda)$ has a one-dimensional kernel at $((1, \mathbf{0}), 0)$ given that $(\mathcal{M}, g_0)$ has no conformal Killing fields.

- Use metric g_0 to build one-parameter family g_λ with no conformal Killing fields
- Verify that the operator $F((\phi, \mathbf{w}), \lambda)$ is analytic in a neighborhood of $((1, \mathbf{0}), 0)$.
- Apply a Liapunov-Schmidt Reduction to obtain the following solution curve in a neighborhood of $((1, \mathbf{0}), 0)$:

$$(\phi(s), \mathbf{w}(s)) = (s+1)\hat{v}_0 + \psi((s+1)\hat{v}_0, \gamma((s+1)\hat{v}_0)) \qquad (16)$$
$$\lambda(s) = \gamma((s+1)\hat{v}_0).$$

The curve in (16) is analytic for $s \in (-\delta, \delta)$, where $\delta > 0$

Analysis of $\lambda(s)$

- In unscaled CTT case, we took a Taylor expansion of $\gamma(s)$ and $f(s) = \psi((s+1)\hat{v}_0, \gamma((s+1)\hat{v}_0))$ and conducted an analysis to determine which lower order terms were nonzero. In particular, we were able to show that $\ddot{\lambda}(0) \neq 0$ was the first nonzero coefficient in Taylor series of $\lambda(s)$, which implied our non-uniqueness result in the unscaled case.
- Scaled CTT case is not as amenable to this analysis. It is unclear what the first nonzero term in Taylor expansion of $\lambda(s)$ is.
- Can use analyticity of $\lambda(s)$ and positive Yamabe, far-from-CMC solution theory in [6] to draw some conclusions about $((\phi(s), \mathbf{w}(s), \lambda(s))$.

Properties of $\lambda(s)$

The function $\lambda(s)$ has the following three important properties:

- Solution theory in [6] implies that for $\lambda_0 > 0$ sufficiently small, scaled CTT equations have a solution. Therefore, for $\lambda_0 > 0$ sufficiently small, there exists $s_0 > 0$ such that $\lambda_0 = \lambda(s_0)$.
- By construction, $\lambda(0) = 0$.
- There is no subinterval $I \subset (-\delta, \delta)$ such that $\lambda(s) = 0$ for each $s \in I$ (follows from analyticity of $\lambda(s)$).

The above three properties imply that either there exists $\lambda_0 < 0$ and s_0 such that $\lambda(s_0) = \lambda_0$ or that there exists $\lambda_1 > 0$ and s_1, s_2 such that $\lambda_1 = \lambda(s_1) = \lambda(s_2)$.

Possible Behavior of Solution Curve

The above analysis implies that for given data $(g_0, \tau, \sigma, \rho, j^a)$, where $R(g_0) = 0$ and g_0 has no conformal Killing fields, we can always find a metric g_{λ_0} in any neighborhood of g_0 for which one of the following holds:

- $R(g_{\lambda_0}) = \lambda_0 < 0$ and the scaled CTT equations with specified data $(g_{\lambda_0}, \lambda_0^2\tau, \lambda_0^2\sigma, \lambda_0^2\rho, \lambda_0^2 j^a)$ have a solution $(\phi, \mathbf{w})$.
- $R(g_{\lambda_0}) = \lambda_0 > 0$ and solutions to the scaled CTT equations with specified data $(g_{\lambda_0}, \lambda_0^2\tau, \lambda_0^2\sigma, \lambda_0^2\rho, \lambda_0^2 j^a)$ are non-unique.

Final Remarks

The following are are interesting questions regarding the non-uniqueness analysis of the scaled CTT equations.

- What is the first non-zero term in the Taylor expansion of $\lambda(s)$?
- What effect does τ have on the above analysis?
- How can one rigorously construct metrics satisfying $R(g_0) = 0$ with no conformal Killing fields?
- What affect does the size of σ have on the solution theory of the constraints?

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