Title: Gross-Zagier Formula: Why Is It Right?

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Date: 2014/08/18 Time: 9:30 am

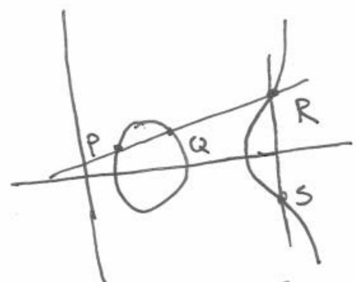
Elliptic curve / $\mathbb{Q}$

$$y^2 = x^3 + ax + b \qquad 4a^3 + 27b^2 \neq 0$$

$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus$ finite grp

$\qquad\qquad E(\mathbb{Q})_{tor}$

Mordell-Weil Thm



$P + Q = S$

$\# E(\mathbb{Q})_{tor} < 16$

$r=0$ for 50% of E with certain order

$r=1$ for 50% ...

Thm (Bhargava - Bhargava - Skinner - Zhang)

$r=0$ for at least "15%" of E

$r=1$ for at least "30%" of E

Question How to construct points in $E(\mathbb{Q})$?

X variety, $\quad E_0(X) = \mathrm{Hom}(X, E)$

$\qquad\qquad\qquad\qquad p \mapsto 0$

$$E_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathrm{Hom}_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}(T_p(E), T_p(\mathrm{Jac}(X)))$$

Tate's conj. Faltings (1983)

$X = {}_{\Gamma}\backslash \mathcal{H} \sqcup \mathbb{P}^1(\mathbb{Q})$ Riemann surface

$\Gamma \hookrightarrow SL_2(\mathbb{Z})$ congruent subgroup

X has canonical model $\mathbb{Q}^{\Gamma} \hookrightarrow \mathbb{Q}^{ab}$

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & \operatorname{Spec} \mathbb{Q}^{\Gamma} \arrow[r] & \operatorname{Spec} \mathbb{Q} \end{tikzcd}$$

$E_0(X) \otimes \mathbb{Q} = \mathrm{Hom}_0(X, E) \otimes \mathbb{Q}$

Cusps $\to$ torsion points

$E_0(X) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{Hom}(V_p(E), V_p(\mathrm{Jac}(X)))$

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$\neq$ (Wiles, BCDT, Taylor-Wiles)

1995-2000?

Heegner points

Fix $K = \mathbb{Q}(\sqrt{d}) \hookrightarrow \mathbb{C}$, $d<0$, $f \in \mathrm{Hom}_0(X, E)$

$CM_K = {}_{\Gamma}\backslash K \cap \mathcal{H} \hookrightarrow X(K^{ab}) \xrightarrow{f} E(K^{ab})$

CM-theory $\qquad \circlearrowleft \mathrm{Gal}(K^{ab}/K)$

$\chi : \mathrm{Gal}(K^{ab}/K) \longrightarrow \mathbb{C}^{\times}$

finite character

$p \in CM_{\Gamma}$

$f \in E_0(X)$

$$P(f,\chi) = \int_{\mathrm{Gal}(K^{ab}/K)} f(p)^{\sigma} \chi(\sigma)\, d\sigma$$

$$= \frac{1}{d} \sum_{\sigma \in \mathrm{Gal}(L/K)} f(p)^{\sigma} \chi(\sigma) \qquad K \subset L \hookrightarrow K^{ab}, \ d = [L:K]$$

$\in E(K^{ab}) \otimes \mathbb{C}$

Question: When is $P(f,\chi) \neq 0$?

Heegner (1952)

$E : y^2 = x^3 - x$ CM by $\mathbb{Q}(i)$

conductor is 32

He constructed ~~an~~ explicit

① $X(8) \longrightarrow E$

$$n = \begin{cases} p & \equiv 5, 7 \bmod 8 \\ 2p & \equiv 6 \bmod 8 \end{cases}$$

$K = \mathbb{Q}(\sqrt{-n})$

2 CM points $\quad p \in CM_{\Gamma(8)}$

Thm $P(f, \chi_0) = 0$

trivial char

$P(f, \chi_0) \in E(K)$

$= E(\mathbb{Q}) \oplus E_n(\mathbb{Q})$

$E_n : ny^2 = x^3 - x$

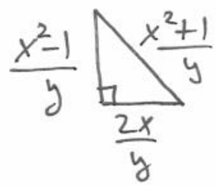
$E(\mathbb{Q}) = E(\mathbb{Q})_{tor}$

Consequence:

$rk\ E_n(\mathbb{Q}) > 0$

Equivalently $\quad n =$ congruent #

$=$ area of Pythagorean triangle



Gross-Zagier formula

Assumption
Heegner Condition
called by Birch

1) $X = X_0(N) \qquad N = \mathrm{cond}(E)$

2) $(d, 2N) = 1 \qquad K = \mathbb{Q}(\sqrt{d})$

$p \mid N \Rightarrow p$ split in K

3) χ unramified

Thm $\quad P(f, \chi) \neq 0$ iff

$L'(E, \chi, 1) \neq 0$

RMK:

1) $L(E, \chi, s) :=$ Rankin-Selberg convolution of $L(E, s)$ and $L(\chi, s)$

2) $P(f, \chi)$ is defined using a canonical CM point on $X_0(N)$

$$X_0(N)(\mathbb{C}) = \left\{ \begin{array}{l} (E_1 \xrightarrow{\phi} E_2) \\ \ker\phi \simeq \mathbb{Z}/N\mathbb{Z} \end{array} \right\}$$

$L(E, \chi, s) = \sum_{\mathfrak{a} \hookrightarrow \mathcal{O}_K} \frac{a_{\mathfrak{a}} \chi(\mathfrak{a})}{N(\mathfrak{a})^s}$ hol continuation to whole complex plane

$L(E, s) = \sum \frac{a_n}{n^s}$

Heegner Conditions $\Rightarrow$ $L(E,\chi,s)$

$$= -L(E,\chi,2-s)$$

$$L(E,\chi,1) = 0$$

$$P = \left(\mathbb{C}/\mathcal{O}_K \longrightarrow \mathbb{C}/\mathfrak{a}^{-1} \right)$$

$\mathfrak{a} \hookrightarrow \mathcal{O}_K \qquad \mathcal{O}_K/\mathfrak{a} \simeq \mathbb{Z}/N\mathbb{Z}$

existence of $\mathfrak{a}$ $\Leftarrow$ Heegner condition

<u>Gross-Zagier formula</u> $\quad u = \frac{1}{2} \# \mathcal{O}_K^\times$

$$\langle P(f,\chi), P(f,\chi) \rangle_{NT} = \frac{h_K\, u^2 \sqrt{|d|}}{8\pi^2 \underbrace{(f,f)}_{=\deg f}} L'(f,\chi,1)$$

$f: X_0(N) \longrightarrow E$

$\langle\,,\,\rangle_{NT}$ Neron-Tate Height pairing