MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

# The Pro-Étale Site - Aise Johan de Jong

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**Summary**: In this lecture the speaker introduces the pro-étale site on a locally Noetherian adic space. We discuss the category-theoretic definition of this site, and see what sort of data we must deal with to make sense of it. Basic properties of the site are reviewed, with some examples to show how the proofs have to go. Finally, we see how to use this site, especially for perfectoid spaces, to compute étale cohomology.

Let $X$ be a locally Noetherian adic space. This has an étale site $X_{\text{ét}}$; we want to define a *pro-étale site* $X_{\text{proét}}$ as a full subcategory of pro-$X_{\text{ét}}$.

**Definition 1.** The category pro-$X_{\text{ét}}$ is defined by having objects consisting of directed inverse systems in $X_{\text{ét}}$. For morphisms, if $U = (U_i)$ and $V = (V_j)$ of two pro-objects are given by

$$\lim_i \operatorname{colim}_j \operatorname{Mor}(U_i, V_j) = \varprojlim_j \varinjlim_i \operatorname{Mor}(U_i, V_j).$$

Notation: If $U = (U_i)$ then define $|U| = \lim_i |U_i|$ as the "underlying topological space". An open subset of this is a union of open subsets of the $U_i$.

**Definition 2.** An object $U$ of pro-$X_{\text{ét}}$ is in $X_{\text{proét}}$ if and only if we can write it as (i.e. can find an object isomorphic to it in the category) $U = (U_i)_{i \in I}$ (for some index set $I$) such that

1. The base morphisms $U_i \to X$ are all étale.
2. The transition maps $U_i \to U_j$ are surjective finite étale.

So think of a pro-étale object as being a system directed by $\mathbb{N}$ with the map down to $X$ being étale and the transition maps being finite étale! Now, need to make this into a site.

**Definition 3.** A family of morphisms $\{f^t : U^t \to U\}$ in the category $X_{\text{proét}}$ is a *covering* if

1. $f^t$ satisfies properties (1) and (2) of the previous definition translated to pro-language, i.e. $f^t$ is a *pro-étale morphism.*

2. $|U| = \bigcup f^t(|U^t|)$.

Baby example: Let $U = (U_i)_{i \geq 0}$ be an object of $X_{\text{proét}}$ (so each $U_i \to U_j$ is finite étale), and we're given étale morphisms $W_{n,n} \to U_n$ étale. Then can take $W_{n,k} = W_{n,n} \times_{U_n} U_k$ whenever $k \geq n$. So the system $W^n = (W_{n,k})_{k \geq n}$ is an object in $X_{\text{proét}}$. Then have a natural morphism $f^n : W^n \to U$ in $X_{\text{proét}}$. Finally, we can conclude that $\{f^n\}$ is a covering iff $\coprod |W^n|$ surjects to $|U|$.

**Lemma 4.** *We have:*

- $X_{\text{proét}}$ *is a site.*
- *Pro-étale morphisms are open (on the level of underlying topological spaces).*
- $X_{\text{proét}}$ *has all finite limits.*
- *If* $U \in X_{\text{proét}}$ *and* $W \subseteq |U|$ *is quasicompact open then* $W = |V|$ *for some* $V \to U$ *in* $X_{\text{proét}}$.

A lot of the arguments here are formal so we won't do them, but let's get an idea of how by proving existence of equalizers in a baby case. Even in the baby case we'll see it's useful to know $X$ is locally Noetherian (which we haven't used so far). Suppose we have objects $U, V$ indexed by the natural numbers and $a, b : U \to V$ two morphisms. Further suppose that $a, b$ actually have morphisms $a_i, b_i : U_i \to V_i$, such that all of the squares in the following diagrams commute (and recall that the vertical maps are finite étale surjective maps).

$$
\begin{array}{ccc}
\cdots & & \cdots \\
\downarrow & & \downarrow \\
U_2 & \overset{a_2}{\underset{b_2}{\rightrightarrows}} & V_2 \\
\downarrow & & \downarrow \\
U_1 & \overset{a_1}{\underset{b_1}{\rightrightarrows}} & V_1 \\
\downarrow & & \downarrow \\
U_0 & \overset{a_0}{\underset{b_0}{\rightrightarrows}} & V_0
\end{array}
$$

How do we form equalizers? Well, we can start by forming the equalizers

$E_i \to U_i$ of $a_i, b_i : U_i \to V_i$, getting another column in the tower.

$$
\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
E_2 & \longrightarrow & U_2 & \overset{a_2}{\underset{b_2}{\rightrightarrows}} & V_2 \\
\downarrow & & \downarrow & & \downarrow \\
E_2 & \longrightarrow & U_1 & \overset{a_1}{\underset{b_1}{\rightrightarrows}} & V_1 \\
\downarrow & & \downarrow & & \downarrow \\
E_0 & \longrightarrow & U_0 & \overset{a_0}{\underset{b_0}{\rightrightarrows}} & V_0
\end{array}
$$

Further assume everything is affinoid, so it's quasi-compact and has finitely many components. Now, for $k \geq n$ set $E_{n,k}$ to be the image of $E_k \to U_n$. Each $E_{n,k}$ is open and closed in $U_n$ (because we have finite étale morphisms in the tower). Since $U_n$ has finitely many connected components, the chain $E_{n,k} \supseteq E_{n,k+1} \supseteq \cdots$ stabilizes, so we can take $E_{n,\infty} = \bigcap_{k \geq n} E_{n,k}$ and get $(E_{n,\infty})$ is in $X_{\text{proét}}$ and is the equalizer.

Now, suppose $f : X \to Y$ is a morphism of locally Noetherian adic spaces. Then you get a commutative diagram of functors of sites

$$
\begin{array}{ccc}
X_{\text{proét}} & \longleftarrow & Y_{\text{proét}} \\
\uparrow & & \uparrow \\
X_{\text{ét}} & \longleftarrow & Y_{\text{ét}}
\end{array}
$$

and then a diagram of functors of topoi

$$
\begin{array}{ccc}
X^{\sim}_{\text{proét}} & \xrightarrow{f_{\text{proét}}} & Y^{\sim}_{\text{proét}} \\
{\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_Y} \\
X^{\sim}_{\text{ét}} & \xrightarrow[f_{\text{ét}}]{} & Y^{\sim}_{\text{ét}}.
\end{array}
$$

**Lemma 5.** *If $U = (U_i)$ is in $X_{\text{proét}}$ and is quasicompact and quasiseparated; then we have*

$$H^q(U, \nu^* \mathcal{F}) = \operatorname{colim} H^q(U_i, \mathcal{F}).$$

Proof of this is purely formal; use the Čech-to-cohomology spectral sequence.

**Lemma 6.** *We have $\nu^* R f_{\text{ét}*} = R f_{\text{proét}*} \nu^* \mathcal{F}$.*

Another example: Let $X = \mathrm{Spa}(K, K^+$ for $K$ a nonarchimedean field and $K^+ = K^\circ$. Then $X_{\text{proét}} = X_{\text{profét}}$, which consists of profinite sets $S$ with continuous $G_K$-actions. A family $\{f^t : S^t \to S\}$ is a covering iff the $f^t$ are open and jointly surjective. Even if $K = \overline{K}$, this is interesting (i.e. doesn't just give us the category of sheaves on a point). Remark: In this example, Scholze shows

$$H^i(X_{\text{proét}}, \varprojlim \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = H^i_{\text{cont}}(G_K, \mathbb{Z}_\ell).$$

From now on, work over a perfectoid field $K$ of characteristic zero, $K^+$ an open and bounded valuation subring, and $X \to \operatorname{Spa}(K, K^+)$.

**Definition 7.** $U \in X_{\text{proét}}$ is *affinoid perfectoid* if $U \cong (U_i)$ with $U_i = \operatorname{Spa}(R_i, R_i^+)$ such that $(R, R^+)$ is perfectoid affinoid where $R^+ = (\operatorname{colim} R_i^+)^\wedge$ and $R = R^+[1/p]$. If so then $\operatorname{Spa}(R, R^+) \sim \lim U_i$.

Example: $\mathbb{T}^n = \operatorname{Spa}(K\langle T_i^{\pm 1}\rangle, K^+\langle T_i^{\pm 1}\rangle)$. Then the tower with multiplication-by-$p$ maps

$$\cdots \to \mathbb{T}^n \to \mathbb{T}^n \to \mathbb{T}^n$$

is affinoid perfectoid.

**Lemma 8.** *Suppose that $U = (U_i)$ is affinoid perfectoid as in the definition of above. Further suppose we have a finite étale map $V_{i_0} \to U_{i_0}$ with $U_{i_0}$ a rational subset. Then $V = (U_i \times_{U_{i_0}} V_{i_0})_{i \geq i_0}$ is affinoid perfectoid.*

**Corollary 9.** *If $X \to \operatorname{Spa}(K, K^+)$ is smooth, then every object of $X_{\text{proét}}$ has a covering by affinoid perfectoids.*

The proof follows because since $X$ is smooth, it's locally étale over $\mathbb{T}^n$, and we have a covering of $X$ in that case. There's also an argument of Colmez which proves this for general locally Noetherian adic spaces over $K$.

Contractible objects: Suppose $X = \operatorname{Spa}(A, A^+)$ is an affinoid Noetherian adic space over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Then there are lots of $U \in X_{\text{proét}}$ such that $H^i(U, \underline{\mathbb{F}_p}) = 0$ for all $i > 0$. In fact, if $X$ is connected then we have a much stronger statement

$$H^i_{\text{cont}}(\pi_1(X, \overline{x}), \mathbb{F}_p) \cong H^i(X_{\text{ét}}, \mathbb{F}_p).$$