Equidistribution of expanding translates of curves on homogeneous spaces and Diophantine approximation

Joint with Nimish Shah

Lei Yang [1]

[1]Mathematical Sciences Research Institute

Advances in homogeneous dynamics, 2015

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation
2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation
2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

Expanding horospherical subgroups and equidistribution

- Let G be a semisimple Lie group, and let Γ be a lattice of G. Then G/Γ admits a unique probability G-invariant measure, denoted by μ_G. Fix a diagonalizable one parameter subgroup $A = \{a(t) : t \in \mathbb{R}\} \subset G$, and let U_G^+ denote the expanding horospherical subgroup of the positive direction of A in G, i.e.,

$$U_G^+ := \{g \in G : a(-t)ga(t) \to e \text{ as } t \to +\infty\}.$$

Expanding horospherical subgroups and equidistribution

- Let G be a semisimple Lie group, and let Γ be a lattice of G. Then G/Γ admits a unique probability G-invariant measure, denoted by μ_G. Fix a diagonalizable one parameter subgroup $A = \{a(t) : t \in \mathbb{R}\} \subset G$, and let U_G^+ denote the expanding horospherical subgroup of the positive direction of A in G, i.e.,

$$U_G^+ := \{g \in G : a(-t)ga(t) \to e \text{ as } t \to +\infty\}.$$

- Take an open subset $\Omega \subset U_G^+$ and a point $x = g\Gamma \in G/\Gamma$, it is well known that the expanded translates $\{a(t)\Omega x : t > 0\}$ of Ωx by $\{a(t) : t > 0\}$ tend to be equidistributed in G/Γ, as $t \to +\infty$. This follows from mixing of the action of A (Margulis' thesis).

Curves in horospherical subgroups

- One could ask the following finer question: if Ω is a piece of a curve in U_G^+, does the same equidistribution result hold?

Curves in horospherical subgroups

- One could ask the following finer question: if Ω is a piece of a curve in U_G^+, does the same equidistribution result hold?
- Mixing of the action of A is insufficient for this problem.

General setting of the problem

Let H be a semisimple Lie group. Fix a diagonalizable one parameter subgroup $A = \{a(t) : t \in \mathbb{R}\} \subset H$. Let G be a Lie group containing H, and let Γ be a lattice of G.
Let

$$\varphi : I = [a, b] \to U_H^+$$

be a piece of analytic curve in U_H^+. Given a point $x = g\Gamma \in G/\Gamma$, Ratner's Theorem tells that the closure of Hx is a homogeneous subspace Fx, where F is a Lie subgroup of G containing H. One can ask whether the expanded curves $\{a(t)\varphi(I)x : t > 0\}$ tend to be equidistributed in Fx.

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation
2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

Related results

- [Shah, 2009, Duke Math. Journal]: $H = \mathrm{SO}(n, 1)$, $A = \{a(t) : t \in \mathbb{R}\}$ is a Cartan subgroup of H, $G = \mathrm{SO}(m, 1)$. It is proved that if under the natural visual map $\mathrm{Vis} : \mathrm{SO}(n, 1) \to \partial\mathbb{H}^n$, the image of the curve is not contained in a proper subsphere of $\partial\mathbb{H}^n$, then the equidistribution holds.

Related results

- [Shah, 2009, Duke Math. Journal]: $H = \mathrm{SO}(n,1)$, $A = \{a(t) : t \in \mathbb{R}\}$ is a Cartan subgroup of H, $G = \mathrm{SO}(m,1)$. It is proved that if under the natural visual map $\mathrm{Vis} : \mathrm{SO}(n,1) \to \partial\mathbb{H}^n$, the image of the curve is not contained in a proper subsphere of $\partial\mathbb{H}^n$, then the equidistribution holds.
- [Yang, 2012]: $H = \mathrm{SO}(n,1)$, $A = \{a(t) : t \in \mathbb{R}\}$ is a Cartan subgroup of H, and general Lie group G. It is proved that the above result holds for general G.

Related results

- [Shah, 2009, Inventiones Math.]: $H = \mathrm{SL}(n+1, \mathbb{R})$, $A = \{\mathrm{diag}\{e^{nt}, e^{-t}, \ldots, e^{-t}\} : t \in \mathbb{R}\}$ and general Lie group G. It is proved that if the curve is not contained in a proper affine hyperplane of $U_H^+ = \mathbb{R}^n$, then the equidistribution holds.

Related results

- [Shah, 2009, Inventiones Math.]: $H = \mathrm{SL}(n+1, \mathbb{R})$, $A = \{\mathrm{diag}\{e^{nt}, e^{-t}, \ldots, e^{-t}\} : t \in \mathbb{R}\}$ and general Lie group G. It is proved that if the curve is not contained in a proper affine hyperplane of $U_H^+ = \mathbb{R}^n$, then the equidistribution holds.
- [Shah, 2010, Journal of Amer. Math. Soc.]: $H = \mathrm{SL}(n+1, \mathbb{R})$, G general Lie group, $A = \{\mathrm{diag}\{e^{\sum_{i=1}^{n} \lambda_i t}, e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}\} : t \in \mathbb{R}\}$, and the curve is restricted on the first row (the same as above). In this case, the same result as above holds.

Related results

- [Shah, 2009, Inventiones Math.]: $H = \mathrm{SL}(n+1, \mathbb{R})$, $A = \{\mathrm{diag}\{e^{nt}, e^{-t}, \dots, e^{-t}\} : t \in \mathbb{R}\}$ and general Lie group G. It is proved that if the curve is not contained in a proper affine hyperplane of $U_H^+ = \mathbb{R}^n$, then the equidistribution holds.
- [Shah, 2010, Journal of Amer. Math. Soc.]: $H = \mathrm{SL}(n+1, \mathbb{R})$, G general Lie group, $A = \{\mathrm{diag}\{e^{\sum_{i=1}^n \lambda_i t}, e^{-\lambda_1 t}, \dots, e^{-\lambda_n t}\} : t \in \mathbb{R}\}$, and the curve is restricted on the first row (the same as above). In this case, the same result as above holds.
- [Yang, 2013]: $H = \mathrm{SL}(2n, \mathbb{R})$, $A = \mathrm{diag}\{e^t \mathrm{I}_n, e^{-t} \mathrm{I}_n\}$ and general Lie group G. It is proved that if the curve in U_H^+ satisfies some geometric conditions, then the equidistribution result holds.

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation
2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

Our case

- In this talk, $H = \mathrm{SL}(m+n, \mathbb{R})$,

$$A := \left\{ a(t) := \begin{bmatrix} e^{nt}\mathrm{I}_m & \\ & e^{-mt}\mathrm{I}_n \end{bmatrix} : t \in \mathbb{R} \right\},$$

for $X \in \mathrm{M}(m \times n, \mathbb{R})$, denote

$$u(X) := \begin{bmatrix} \mathrm{I}_m & X \\ & \mathrm{I}_n \end{bmatrix},$$

then

$$U_H^+ = \{u(X) : X \in \mathrm{M}(m \times n, \mathbb{R})\}.$$

G, Γ and $x = g\Gamma \in G/\Gamma$ are general. Without loss of generality, we may assume that Hx is dense in G/Γ.

Our case

- In this talk, $H = \mathrm{SL}(m+n, \mathbb{R})$,

$$A := \left\{ a(t) := \begin{bmatrix} e^{nt}\mathrm{I}_m & \\ & e^{-mt}\mathrm{I}_n \end{bmatrix} : t \in \mathbb{R} \right\},$$

  for $X \in \mathrm{M}(m \times n, \mathbb{R})$, denote

$$u(X) := \begin{bmatrix} \mathrm{I}_m & X \\ & \mathrm{I}_n \end{bmatrix},$$

  then

$$U_H^+ = \{u(X) : X \in \mathrm{M}(m \times n, \mathbb{R})\}.$$

  G, Γ and $x = g\Gamma \in G/\Gamma$ are general. Without loss of generality, we may assume that Hx is dense in G/Γ.
- Considering a curve in U_H^+ is equivalent to considering a curve

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

  in the space of m by n matrices.

Generic condition: $m = n$

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R}).$$

For $m = n$, we say φ is *generic* if there exists a point $s_0 \in I$ such that $\varphi^{'}(s_0)$ has full rank.Then there is a subinterval $J_{s_0} \subset I$ such that $\varphi(s) - \varphi(s_0)$ is invertible for all $s \in J_{s_0}$.

Generic: general case

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R}).$$

- For $m < n$, we rewrite $\varphi(s)$ as $\begin{bmatrix}\varphi_1(s), & \varphi_2(s)\end{bmatrix}$, where $\varphi_1(s)$ is the first m by m block, and $\varphi_2(s)$ is the rest m by $n-m$ block. We say φ is *generic* if there exists a point s_0 and a subinterval $J_{s_0} \subset I$ such that $\varphi_1(s) - \varphi_1(s_0)$ is invertible for $s \in J_{s_0}$; and if we define

$$\psi : J_{s_0} \to \mathrm{M}(m \times (n-m), \mathbb{R})$$

by $\psi(s) = (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$, then ψ is *generic*.

Generic: general case

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R}).$$

- For $m < n$, we rewrite $\varphi(s)$ as $\begin{bmatrix}\varphi_1(s), & \varphi_2(s)\end{bmatrix}$, where $\varphi_1(s)$ is the first m by m block, and $\varphi_2(s)$ is the rest m by $n-m$ block. We say φ is *generic* if there exists a point s_0 and a subinterval $J_{s_0} \subset I$ such that $\varphi_1(s) - \varphi_1(s_0)$ is invertible for $s \in J_{s_0}$; and if we define

  $$\psi : J_{s_0} \to \mathrm{M}(m \times (n-m), \mathbb{R})$$

  by $\psi(s) = (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$, then ψ is *generic*.
- For $m > n$, φ is called *generic* if its transpose

  $$\varphi^{\mathrm{T}} : I = [a, b] \to \mathrm{M}(n \times m, \mathbb{R})$$

  is *generic*.

Main result

Theorem (Nimish Shah and Lei Yang)

Let μ_t denote the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. If $(m,n)=1$, then if an analytic curve $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$ is generic, *then $\mu_t \to \mu_G$ as $t \to \infty$, i.e., $a(t)u(\varphi(I))x$ tends to be equidistributed in G/Γ as $t \to +\infty$.*

Main result

Theorem (Nimish Shah and Lei Yang)

Let μ_t denote the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. If $(m, n) = 1$, then if an analytic curve $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$ is generic, *then $\mu_t \to \mu_G$ as $t \to \infty$, i.e., $a(t)u(\varphi(I))x$ tends to be equidistributed in G/Γ as $t \to +\infty$.*

Remarks:

- For general (m, n), we need to define another geometric condition called *supergeneric*. If the curve is *supergeneric*, then the equidistribution result holds.

Main result

Theorem (Nimish Shah and Lei Yang)

Let μ_t denote the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. If $(m,n)=1$, then if an analytic curve $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$ is generic, *then $\mu_t \to \mu_G$ as $t \to \infty$, i.e., $a(t)u(\varphi(I))x$ tends to be equidistributed in G/Γ as $t \to +\infty$.*

Remarks:

- For general (m,n), we need to define another geometric condition called *supergeneric*. If the curve is *supergeneric*, then the equidistribution result holds.
- In the case $m=1$, the *generic* condition is equivalent to say that the curve is not contained in a proper affine subspace (the same condition as in [Shah, 2009, Inventiones Math.]).

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation
2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

Our result in Diophantine approximation

Theorem (Shah and Yang)

If $(m, n) = 1$*, and an analytic curve*

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is generic*, then almost every point on* $\varphi(I)$*, Dirichlet's Theorem is not improvable.*

Our result in Diophantine approximation

Theorem (Shah and Yang)

If $(m, n) = 1$, and an analytic curve

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is generic, *then almost every point on $\varphi(I)$, Dirichlet's Theorem is not improvable.*

Remark: the correspondence between Diophantine approximation and homogeneous dynamics is due to [Dani, 1985, J. Reine Angew. Math.], [Kleinbock and Margulis, 1998, Annals of Math.] and [Kleinbock and Weiss, 2008, Journal of Modern Dynamics].

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation
2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

Ratner's theorem and Linearization technique

Recall that μ_t denotes the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. Assume some limit measure μ_∞ of $\{\mu_t : t \in \mathbb{R}\}$ is not the Haar measure μ_G.

Ratner's theorem and Linearization technique

Recall that μ_t denotes the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. Assume some limit measure μ_∞ of $\{\mu_t : t \in \mathbb{R}\}$ is not the Haar measure μ_G.

- Shah [Shah, 2009, Duke Math. Journal] proved that after some modification, any limit measure of $\{\mu_t : t > 0\}$ is invariant under some unipotent subgroup W of H.

Ratner's theorem and Linearization technique

Recall that μ_t denotes the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. Assume some limit measure μ_∞ of $\{\mu_t : t \in \mathbb{R}\}$ is not the Haar measure μ_G.

- Shah [Shah, 2009, Duke Math. Journal] proved that after some modification, any limit measure of $\{\mu_t : t > 0\}$ is invariant under some unipotent subgroup W of H.
- This will allow us to apply Ratner's theorem on classification of finite measures invariant under a unipotent flow.

Ratner's theorem and Linearization technique

Recall that μ_t denotes the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$. Assume some limit measure μ_∞ of $\{\mu_t : t \in \mathbb{R}\}$ is not the Haar measure μ_G.

- Shah [Shah, 2009, Duke Math. Journal] proved that after some modification, any limit measure of $\{\mu_t : t > 0\}$ is invariant under some unipotent subgroup W of H.
- This will allow us to apply Ratner's theorem on classification of finite measures invariant under a unipotent flow.
- The *linearization technique* allows us to translate everything to a linear representation V of G, and conclude that there is a nonzero vector $v \in V$, such that

$$u(\varphi(s))v \in V^-(A) + V^0(A).$$

Here the decomposition $V = V^+(A) + V^0(A) + V^-(A)$ is according to the eigenspaces of the action of A.

Outline

1. Introduction
   - Motivation
   - Related results
   - Summary of the result
   - Applications to Diophantine approximation

2. Sketch of the proof
   - Ideas to study the limit measures
   - From algebra to geometry

From algebra to geometry

Assume that $\varphi(s_0) = \mathbf{0}$ and $v \in V^-(A) + V^0(A)$. Let $\varphi(s) = [\varphi_1(s), \varphi_2(s)]$, $\psi(s) = \varphi_1^{-1}(s)\varphi_2(s)$. Denote

$$u'(\psi(s)) := \begin{bmatrix} \mathrm{I}_m & & \\ & \mathrm{I}_m & \psi(s) \\ & & \mathrm{I}_{n-m} \end{bmatrix},$$

$$A' := \left\{ a'(t) := \begin{bmatrix} \mathrm{I}_m & & \\ & e^{(n-m)t}\mathrm{I}_m & \\ & & e^{-mt}\mathrm{I}_{n-m} \end{bmatrix} : t \in \mathbb{R} \right\}.$$

From algebra to geometry

The proof goes as follows:

From algebra to geometry

The proof goes as follows:

- Take the highest eigenspace projection v_μ of v.

From algebra to geometry

The proof goes as follows:

- Take the highest eigenspace projection v_μ of v.
- Prove that
$$u'(\psi(s))v_\mu \in V^-(A') + V^0(A')$$
for all $s \in J_{s_0}$. This follows a basic lemma on $\mathrm{SL}(2,\mathbb{R})$ representations proved by Shah [Shah, 2009, Duke Math. Journal] and direct calculation.

From algebra to geometry

The proof goes as follows:

- Take the highest eigenspace projection v_μ of v.
- Prove that
$$u'(\psi(s))v_\mu \in V^-(A') + V^0(A')$$
for all $s \in J_{s_0}$. This follows a basic lemma on $\mathrm{SL}(2, \mathbb{R})$ representations proved by Shah [Shah, 2009, Duke Math. Journal] and direct calculation.
- Apply induction.

Thank you!