An improved bound on the Hausdorff dimension of Besicovitch sets in \( \mathbb{R}^3 \): Joshua Zahl, 19 May 2017
joint work with Nets Katz

A Besicovitch set in \( \mathbb{R}^n \) is a compact set containing a line segment pointing in every direction.

**Conjecture.** Every Besicovitch set in \( \mathbb{R}^n \) must have Hausdorff dimension \( n \).

**Theorem.** (Katz, Zahl) Every Kakeya set in \( \mathbb{R}^3 \) has \( H \)-dimension \( \geq \frac{5}{2} + \epsilon_0 \), where \( \epsilon_0 > 0 \).

Let \( \delta > 0 \). A tube is a \( \delta \)-neighborhood of a unit line segment.

**Definition.** A set \( \Pi \) of tubes satisfies the Wolff axioms if (1) every tube in \( \Pi \) is in \( B(0,1) \), (2) for every rectangular prism \( R \) of dimensions \( (2,s,t) \), at most \( \delta - 2st \) tubes from \( \Pi \) are contained in \( R \).

**Theorem.** (Katz, Zahl) Let \( \Pi \) be a set of tubes that satisfy the Wolff axiom. If \( |\Pi| = \delta - 2 \), then \( \delta^{\frac{1}{2} - \epsilon_0} \lessapprox |\cup_{T \in \Pi} T| \).

For each of the tubes, let \( Y(T) \). If \( \sum_{T \in \Pi} |Y(T)| > \lambda \) then

\[
\lambda^{\delta^{\frac{1}{2} - \epsilon_0}} \lessapprox |\cup_{T \in \Pi} Y(T)|
\]

(1)

**Theorem.** (Katz, Laba, Tao) Let \( \Pi \) be a set of \( \delta^{-2} \) tubes pointing in \( \delta \)-separated directions. Then,

\[
|\cup_{T \in \Pi} T| \geq \delta^{\frac{1}{2} - \epsilon_1} \quad \text{or} \quad |\cup_{T \in \Pi} N_{\delta^{\frac{1}{2}}(T)}| \geq (\delta^{\frac{1}{2}})^{\frac{1}{2} - \epsilon_1}.
\]

Heisenberg Group: \( \mathbb{H} = \{(x, y, z) \in \mathbb{C}^3, Im(z) = Im(xy)\} \)

If \( a, b \in \mathbb{R}, c \in \mathbb{C} \), the line \((0, w, b) + \mathbb{C}(1, a, \overline{w}) \subset \mathbb{H} \).

Let \( R = \mathbb{F}_p[t]/(t^2) \). If \( a \in R, a = a_1 + a_2t \) where \( a_1, a_2 \in \mathbb{F}_p \).

\[
X = \{(x_1 + x_2t, y_1 + y_2t, z_1 + z_2t)|z_2 = x_1y_2 - y_1x_2\}.
\]

If \( a, b, c, d \in \mathbb{F}_p, ad - bc = 1 \), then \( (a + aat, b + abt) + R(c + act, d + adt)X \).

\( \Pi : \mathbb{R}^3 \rightarrow \mathbb{F}_p^3, \Pi(x) = \mathbb{F}_p^3 \).

**Vague Theorem.** If \( \Pi \) is a counter example to the above theorem, then it is either the Heisenberg or \( SL_2 \) example.
A regulus is a (quadric) surface in $\mathbb{R}^3$ that is doubly ruled by lines.

If $L_1, L_2, L_3$ are skew lines then the union of the lines incident to $L_1, L_2, L_3$ form a regulus.

$H(T_0)$ is the "hairbrush of $T_0$", the set of tubes from $\Pi$ that hit $T_0$.

A regulus strip is a set of the form $N_\delta(Z) \cap N_{\delta^{1\over 2}}(L) \cap B(0, 1)$, where $N_\delta(Z)$ is a regulus and $N_{\delta^{1\over 2}}(L)$ is a line in $Z$.

If $T$ is a $SL_2$ type set $\Pi$ is a disjoint union of $\delta^{-1\over 2}$ sets each of which is contained in a regulus strip.

Lines in $\mathbb{R}^3 \leftrightarrow$ points in $\mathbb{R}^4$.

Tubes in $\mathbb{R}^3 \leftrightarrow$ $\delta$-balls in $\mathbb{R}^4$

If $T$ is a $SL_2$ type counter example then $image(T)$ in $\mathbb{R}^4$ is contained in $N_{\delta^{1\over 2}}(Z(p))$. $P(a, b, c, d) = ad - bc - 1.$