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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rebekah Palmer **Email/Phone:** palmer@temple.edu (443) 621-7023

Speaker's Name: Jon Carlson

Talk Title: Homological algebra and group representations

Date: 02 / Feb / 18 **Time:** 09 : 30 [✓ **am**] / **pm (circle one)**

List 6-12 key words for the talk: representation theory, groups, homology, invariants, rank varieties, stable modules, support varieties

Please summarize the lecture in 5 or fewer sentences: A standard invariant of a module over a ring is its annihilator. Some geometry can be added by considering the collection of all prime ideals or maximal ideals of the ring that contain the annihilator. Using group cohomology, similar structures can be made to give rough classifications of modular representations of finite groups. Going further, it leads to a classifiction of some distinctve structures in the category of modules over the finite group.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Jon Carlson

$\underbrace{\text{An introduction}}_{\text{(gentle, I hope)}}$ to support varieties

02 Feb 18

There are too many modules; you can't write them all down. If you have one operators on a vector space, you can characterize it uniquely via a Jordan canonical form. If you have two operators, then you can't.

1 Invariants of R-modules

What if we were studying commutative algebra?

$k =$ field, algebraically closed
$R = k[x_1, \dots, x_n]$
Spectrum = collection of prime (or maximal) ideals with topology

Hilbert's Nullstellensatz. *A maximal ideal is the kernel of a point evaluation. (Spectrum $= k^n$)*

M an R-module
$J =$ annihilator of $M = \{x \in R \mid xM = 0\}$
$V(M) =$ all maximal ideals that contain J

The problem is that we're NOT doing commutative algebra; we're doing representation theory.

2 Invariants of kG-modules

$k =$ characteristic $p > 0$. Suppose $G = \langle g_1, \dots, g_n \rangle$, $g_i g_j = g_j g_i$, $g_i^p = 1$.
$\cong \left({}^{\mathbb{Z}}/_{p\mathbb{Z}} \right)$

$$\begin{array}{ccc} H^*(G,k) & \underset{\text{essentially}}{\cong} & k[\zeta_1, \dots, \zeta_n] \\ \| & & \\ \operatorname{Ext}^*_{kG}(k,k) & & \end{array}$$

When $p = 2$, this is precisely true. If p is odd, then there's $\deg = 1$ stuff that wind up being nilpotent.

$p = 2$, $n = 1$, $G = \langle g \rangle$. Let $x = g - 1$ in kG. Then $x^2 = g^2 - 1^2 = 1 - 1 = 0$ (because we are in characteristic 2). So in this case,

$$kG = k[x] \Big/ (x^2)$$

For general characteristic,

$$kG = k[x_1, \ldots, x_n] \Big/ (x_1^p, \ldots, x_n^p)$$

Take the free resolution

$$\begin{tikzcd}
\cdots \arrow[r] & kG \arrow[r, "x"] \arrow[d, Rightarrow, "\zeta"] & kG \arrow[r, "x"] \arrow[d, Rightarrow, "\zeta"] & kG \arrow[r, "x"] \arrow[d, Rightarrow, "\zeta"] & kG \arrow[r, "a"] & k \arrow[r] & 0 \\
& kG \arrow[r] & kG \arrow[r] & kG & & &
\end{tikzcd}$$

which gives us a product. Let M be a (finitely generated) kG-module.

$$H^*(G, k) \times \mathrm{Ext}^*_{kG}(M, M) \to \mathrm{Ext}^*_{kG}(M, M)$$

Then $\mathrm{Ext}^*_{kG}(M, M)$ is an $H^*(G, k)$-module.

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & B_{n-1} \arrow[r] & \cdots \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Recall:

- if M, N are kG-modules, then $M \otimes_k N$ is also a kG-module with $g(m \otimes n) = gm \otimes gn$;
- kF is a self-injective ringl
- $k \otimes M \cong M$.

Tensor the above long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & B_{n-1} \otimes M \arrow[r] & \cdots \arrow[r] & B_1 \otimes M \arrow[r] & B_0 \otimes M \arrow[r] & k \otimes M \arrow[r] & 0
\end{tikzcd}$$

$J =$ annihilator in $H^*(G, k)$ of $\mathrm{Ext}^*_{kG}(M, M)$
$V_G(M) =$ maximal ideals in the spectrum of $H^*(G, k)$ that contain J.

Quillen's Theorem.

$$\bigcap_{\substack{E \text{ elementary} \\ \text{abelian}}} \ker\left(\mathrm{res}_{G,E}\left(H^*(G, k)\right)\right)$$

is a nilpotent ideal (thus is contained in every maximal/prime ideal). Furthermore,

$$\textit{spectrum of } H^*(G, k) = V_G(k) = \lim V_E(k)$$

where

$$\begin{array}{rrcl}
\mathrm{res}: & H^*(G, -) & \to & H^*(E) \\
\mathrm{res}^*: & V_G(k) & \leftarrow & V_E(k)
\end{array}$$

3 Rank Variety

Assume G elementary abelian, $kG = k[x_1, \dots, x_n] \Big/ (x_1^P, \dots, x_n^p)$.

Given a maximal ideal in $H^*(G, k) \sim (\alpha_1, \dots, \alpha_n)$, let

$$v_\alpha = 1 + \sum \alpha_i x_i \text{ and } v_\alpha^p = 1 + (\alpha_i^p x_i^p) = 1 \text{ where } x_i = g_1 - 1$$

We get a subalgebra generated by v_α:

$$k\langle v_\alpha = k[x] \Big/ x^p$$

We define a <u>rank variety</u> (which is closed because M is finitely generated)

$$V_G^r(M) = \{0\} \cup \{\alpha \mid v_\alpha \text{ does NOT act freely on } M\} \subseteq k^n$$

Theorem. $V_G(M) \underset{\textit{homeomorphic}}{\approx} V_G^r(M)$

Properties of the varieties:

1. $V_G(M) = \{0\} \Leftrightarrow M$ is projective
2. for a short exact sequence $0 \to L \to M \to N \to 0$, we have $V_G(M) \subseteq V_G(L) \cup V_G(N)$
3. $V_G(M \otimes N) = V_G(M) \cap V_G(N)$
4. $\underbrace{\hat{V}_G(M)}_{\text{projectivize}}$ is connected if M is indecomposible

4 Stable module category

The <u>stable module category</u> $\mathbf{Stmod}(kG)$

objects: finitely generated kG-modules

morphisms: $\mathrm{Hom}_{\mathbf{Stmod}(kG)}(M, N) = \mathrm{Hom}_{kG}(M, N) \Big/ P\mathrm{Hom}_{kg}(M, N)$
(quotient out those that factor through a projective)

If P is projective, then so is $M \otimes P$ for any M $\Rightarrow$ $\mathbf{Stmod}(kG)$ has a tensor structure

Triangles in the stable category roughly correspond to exact sequences in the module category. We need a translation operator:

$$\begin{tikzcd} M \arrow[r] & \text{injective hull} \arrow[r] & \Omega^{-1}(M) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] & \Omega^{-1}(A) \end{tikzcd}$$

When we go around the triangle, we get back to A with a translation.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & I \arrow[r] \arrow[d] & \Omega^{-1}(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & \text{pushout} \arrow[r] & \Omega^{-1}(A) \arrow[r] & 0
\end{tikzcd}$$

A full subcategory $\mathcal{C}$ is called <u>thick</u> if it is triangulated (if two out of three objects in a triangle are in the subcategory, then so it the third) and closed under direct summands.

Let V be a closed set in $V_G(k) =$ spectrum of $H^*(G,k)$, and let $\mathcal{C}$ be the (full) collection of all kG-modules M such that $V_G(M) \subseteq V$. Then $\mathcal{C}$ is a thick subcategory. Furthermore, $\mathcal{C}$ is also a tensor ideal; that is, if $M \in \mathcal{C}$ and N anything, then $M \otimes N \in \mathcal{C}$.

If $\mathcal{C}$ is any thick subcategory of $\mathbf{Stmod}(kG)$, let $\mathfrak{X} = \{V \mid V = V_G(M) \text{ for some } M \in \mathcal{C}\}$. Then $\mathcal{C}$ is the collection of all M such that $V_G(M) = \mathfrak{X}$.