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How can you generalize $\mathbb{P}^1 - \{0,1,\infty\}$?

$N=1$

Fix subfield k of $\mathbb{C}$.

$$X = \mathbb{P}^N \setminus \{\underbrace{L_1, \dots, L_{N+2}}_{\text{hyperplanes in general position}}\} \qquad N>1$$

$$\pi_1(X,x) \cong \mathbb{Z}^{N+1}$$

$$\pi_j(X,x) = 0 \qquad 1 < j < N$$

$\pi_N := \pi_N(X,x)$ free, rk 1 $\mathbb{Z}\pi_1$-module

<u>Thm</u>: $\pi_N / I^k \pi_N$ is a mixed Tate motive.

<u>Motivation</u>: conjecturally,

- There is a (tannakian) category of mixed motives $/k$, $\mathcal{MM}(k)$

$$\begin{array}{ccc} Var/k & \longrightarrow & \mathcal{MM}(k) \\ & \searrow & \downarrow \scriptstyle\text{realizn} \\ & & Vec_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} X & \longmapsto & H^i_{mot}(X) \\ & \searrow & \swarrow \\ & H^i_B(X) & \end{array}$$

- Conjectures:

$$\mathrm{Ext}^{\cdot}_{MM(k)}(\mathbb{Q}, \mathbb{Q}(n)) \sim K_{\cdot}(k) \otimes \mathbb{Q}.$$

or $H^{\cdot}(X)(n)$

- there is a tannakian cat of mixed Tate motives with correct Exts when k is a number field. (Voevodsky, Levine, Deligne–Goncharov). Not so easy to construct objects.

- Deligne

$$\mathbb{Q}\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, x) / I^n$$

are in $MTM(k)$. $x \in k^{\times} - \{1\}$.

Gives uniform construction of MZV's when $x = \partial/\partial w \in T_1 \mathbb{P}^1$.

Nori's Category of Motives

There is an abelian category $EHM(k)$ effective homological motives.

Do not know how Ext-groups in this category are related to K-theory.

<u>Construction:</u>

① $D \xrightarrow{\text{functor}} R\text{-mod}$ $\qquad R = \text{ring}$

diagram
(graph, quiver)

Associated to this is a universal abelian category

$$\begin{tikzcd}
D \arrow[r] \arrow[rr, bend left] \arrow[dr] & \mathcal{U}(D) \arrow[r, dashed, "\text{get morphism}"] \arrow[d, "ff"] & B \arrow[dl, bend left] \\
& R\text{-mod} &
\end{tikzcd}$$

another factorization

(2) (X, Y, i) $\qquad$ "$H_i(X, Y)$

$\downarrow$

(Y, Z, i)

Nori–Beilinson complex:

Every X has a filtration

$$\cdots \subseteq X_j \subseteq X_{j+1} \subseteq \cdots \qquad \Big| \text{ in } Var_k$$

s.t. $H_\bullet(X_{j+1}, X_j)$ concentrated in degree j. Resulting complex computes $H_\bullet(X)$.

a variety

Philosophy: If X is nilpotent, then we expect $\pi_n(X, x)$ to be motivic.

<u>Thm</u>: If (X, x) is nilpotent $\Rightarrow \pi_n \otimes \mathbb{Q}$ is motivic for all n.

<u>Thm</u> (Patel) If (X, x) is N-nilpotent, then $\pi_N / \pi_N \cdot I^k$ is motivic all k.

Want to give motivic construction. Issue is that Bar construction is not Hopf algebra:

Toy example

$$(TV, \Delta) \qquad \mathbb{L}(V) = PT(V)$$

Instead, have $\mathbb{Z}\Sigma_n$ action

$$s_n = (1-(1,2))(1-(1,2,3)) \dots (1-(1,\dots,n))$$

$$\in \mathbb{Z}\Sigma_n \qquad s_n^2 = n s_n.$$

$p: V^{\otimes n} \longrightarrow \mathbb{L}_n(V)$ proj assoc to s_n.

So don't need coproduct to go from

$T(V)$ to $\mathbb{L}(V)$.

Also: Alexander modules are motivic: completed

$\pi \supseteq [\pi, \pi]$.

$I \subseteq \mathbb{Q}H_1 \curvearrowright H_1([\pi,\pi], \mathbb{Q}) =: V$

aug. ideal

$V/I^k V$ are motivic

(cf. Anderson et al)

Semi-abelian varieties (joint with Nori)

eg $\pi_N / I^k \pi_N \cong H_N(X, \mathbb{Z}\pi/I^k)$

$H_N(X, \mathbb{Z}\pi)$ is $\mathbb{Z}\pi$-module

Remark:

$H^N(X, \mathbb{Z}[\pi]) \cong H^{N+1}(\mathbb{G}_m^{n+1}, H; \mathbb{Z}[\pi])$

Lefschetz

$(x_0, \dots, x_n)$

eg: $x_0 + \dots + x_n = 1$

(have family of generic hyperplanes H)

$(X, Y, f) \qquad f : X \to \mathbb{G}_m^n$

Fix a semi abelian variety $\mathcal{A}$ — eg $\mathbb{G}_m^n$

objects: (X, Y, f) ; $f : X \to \mathcal{A}$

Betti realiz^n : $H^\cdot(X, Y; f^* \mathbb{Z}[\pi]) \qquad \pi = \pi_1(\mathcal{A})$

$\rightsquigarrow \quad SA(R)$

Hodge realiz^n ?

De Rham realiz^n :

<u>Ex</u>: X smooth $/k$, $\mathcal{A} = \mathbb{G}_m$

$X[s] := X \times_k \mathbb{A}^1 \qquad s =$ coord in $\mathbb{A}^1$

$\mathcal{O}_{X[s]} e(s) \xrightarrow{D} \Omega^1_{X[s]/k[s]} e(s)$

$D = \left(d + f \cdot s \frac{df}{f}\right) e(s)$

"Aomoto complex".

$H^\cdot(\Gamma(\text{this complex}))$ is a

$\underbrace{\mathbb{Z}[\pi] \otimes k[s]}_{\text{twisted } \otimes \text{ prod}}$ module

DR realizn lives over $\mathbb{A}^1_s$

$\downarrow$ exp.

Betti " " " $\mathbb{G}_m$

Allows one to compute periods. These live in

of completion

$$k[\![s]\!] \backslash \mathbb{C}[\![s]\!] / \mathbb{Z}[t, t^{-1}]$$

Example: $X = \mathbb{G}_m - \{1\}$, $Y = \mathbb{C}$

$\rightsquigarrow$ $H^1(\mathbb{G}_m - \{1\}, C, \mathbb{Z}[\pi])$

$\rightsquigarrow$ get polylogs.