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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Lars Hesselholt

Talk Title: K-theory of division algebras over local fields

Date: 3 / 29 / 19 **Time:** 11 : 00 [✓ am] / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:
Using higher-categorical characterisations, they compute K-theory of division algebras over local fields, at least when p>2.

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K-THEORY OF DIVISION ALGEBRAS OVER LOCAL FIELDS

LARS HESSELHOLT

Joint with Michael Larsen and Ayelet Lindenstrauss.

Thanks to higher algebra we can get invariants through *definitions* rather than just *constructions*.

Blumberg-Gepner-Tabuada: construct a functor $\mathcal{U}_{\text{loc}} \colon \mathbf{Cat}_\infty^{\text{st}} \to \text{NMot}$ (called Z in the lecture, but this is how BGT calls it) with codomain the infinity category of small stable $(\infty, 1)$-categories and output some category such that we have the following universal properties:

1. NMot is a stable infinity category.
2. If $f \colon \mathcal{C} \to \mathcal{D}$ is an equivalence after idempotent completion (i.e. a Morita equivalence), then $\mathcal{U}_{\text{loc}}(f)$ is an equivalence.
3. If we have a bicartesian square in $\mathbf{Cat}_\infty^{\text{st}}$ (which itself is not stable, so bicartesian means something):

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}'' \end{tikzcd}$$

   then $\mathcal{U}_{\text{loc}}$ sends this to a bicartesian square, i.e. a triangle $\mathcal{U}_{\text{loc}}(\mathcal{C}') \to \mathcal{U}_{\text{loc}}(\mathcal{C}) \to \mathcal{U}_{\text{loc}}(\mathcal{C}'')$.

Using the fact that $\mathbf{Cat}_\infty^{\text{st}}$ has a monoidal structure [Lurie], we can upgrade $\mathcal{U}_{\text{loc}}$ to a monoidal functor. The monoidal unit $\mathbb{1}$ in $\mathbf{Cat}_\infty^{\text{st}}$ is $\text{Perf}(\mathbb{S})$, i.e. $D(\mathbb{S})^\omega$ the compact objects in the derived category of $\mathbb{S}$-modules.

Theorem 1 (BGT)**.** (Theorem/Definition) $K(\mathcal{C}) \simeq \underline{\text{Map}}_{\text{NMot}}(\mathcal{U}_{\text{loc}}(\mathbb{1}), \mathcal{U}_{\text{loc}}(\mathcal{C}))$. That is, nonconnective K-theory is representable.

As a particular case, if X is a quasicompact quasiseparated scheme, then $\text{Perf}(X) \in \mathbf{Cat}_\infty^{\text{st}}$ so we can get its K-theory via the above machine. The category Nmot is

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Notes by Ian Coley.

mysterious per se, although it's great for this definition. We can try to understand it via other additive invariants: Nikolaus-Scholze came up with the following schematic:

$$\begin{tikzcd}
\mathbf{Cat}_\infty^{\mathrm{st}} \arrow[rr, "\mathcal{U}_{\mathrm{loc}}"] \arrow[dr, "THH"'] & & \mathrm{NMot} \arrow[dl, dashed, "\exists!"] \\
& \mathrm{CycSp} \arrow[d, "f=\mathrm{forget}"'] & \\
& \mathrm{Sp}^{B\mathbb{T}} &
\end{tikzcd}$$

where THH is topological Hochschild homology, CycSp is the category of cyclotomic spectra, and $\mathrm{Sp}^{B\mathbb{T}}$ is the category of spectra with a genuine S^1-action. There exists a unique trace map from NMot to CycSp by the universal property. We can understand this more explicitly for some $\mathcal{C} \in \mathbf{Cat}_\infty^{\mathrm{st}}$:

$$\begin{tikzcd}
K(\mathcal{C}) \simeq & \underline{\mathrm{Map}}_{\mathrm{NMot}}(\mathcal{U}_{\mathrm{loc}}(\mathbb{1}), \mathcal{U}_{\mathrm{loc}}(\mathcal{C})) \arrow[d, "\mathrm{tr}"'] \\
TC(\mathcal{C}) \simeq & \underline{\mathrm{Map}}_{\mathrm{CycSp}}(THH(\mathbb{1}), THH(\mathcal{C})) \arrow[d, "\mathrm{forget}"'] \\
TC^{-}(\mathcal{C}) \simeq & \underline{\mathrm{Map}}_{\mathrm{Sp}^{B\mathbb{T}}}(f(THH(\mathbb{1})), f(THH(\mathcal{C}))) \arrow[d, shift right, "\mathrm{can}"'] \arrow[d, shift left, "\prod_p \phi_p"] \\
TP(\mathcal{C})^{\wedge} \simeq & \prod_p TP(\mathcal{C})^{\wedge}_p
\end{tikzcd}$$

where TC is the equalizer of the two parallel arrows from TC^{-} to $TP^{\wedge}$. This is all 'noncommutative' since it only depends on the category $\mathcal{C}$, hence we also get $TC(X)$ and $TP(X)^{\wedge}$ as invariants.

Now, fix K a local field, D/K a division algebra of finite index. Let π be a uniformizer with $D \supset K(\pi)/K$ a totally ramified extension of degree d, and $D \supset L/K$ the maximal unramified subfield splitting D, with deg $= d$. We can choose π such that $\pi^d \in \mathcal{O}_K$, so we get an analogous picture with $\mathcal{O}_D \supset \mathcal{O}_K[\pi] \supset \mathcal{O}_K$ and $\mathcal{O}_D \supset \mathcal{O}_L \supset \mathcal{O}_K$ with $\mathcal{O}_L$ étale over $\mathcal{O}_K$. We can simultaneously choose π such that $\sigma\colon D \to D$ given by conjugation by π restricts to an automorphism of L that generates the (cyclic) Galois group $G = \mathrm{Gal}(L/K)$. We then get an isomorphism $G \to \mathrm{Gal}(k_L/k_K)$ the Galois group of the residue fields, so we must have $\sigma \mapsto \phi^r$ for some power r of Frobenius which is coprime to d. That gives us the Hasse invariant $[D] \in \mathrm{Br}(K) \mapsto r/d \in \mathbb{Q}/\mathbb{Z}$.

Okay, now we can begin our setup: consider the diagram:

$$\begin{tikzcd}
D \arrow[r, "\mathrm{id}\otimes f"] & D \otimes_K L \\
& L \otimes_K L \arrow[u, "\pi"'] \arrow[d, "\delta=\text{multiplication}"] \\
K \arrow[r, "f"'] & L
\end{tikzcd}$$

Passing to perfect complexes, we get some additional adjoints:

$$\begin{tikzcd}
\mathrm{Perf}\, D \arrow[r, "f^*"] & \mathrm{Perf}\, D \otimes_K L \arrow[d, shift right, "\pi_*"'] \\
& \mathrm{Perf}\, L \otimes_K L \arrow[u, shift right, "\pi^!"'] \arrow[d, shift right, "\delta^*"'] \\
\mathrm{Perf}\, K \arrow[r, "f^*"'] & \mathrm{Perf}\, L \arrow[u, shift right, "\delta_*"']
\end{tikzcd}$$

Call that left adjoint the reduced trace $\mathrm{Trd}_{D\otimes_K L/L}$ and the right adjoint $\mathrm{Ird}_{D\otimes_K L/L}$. These are actually adjoint equivalences.

In the above picture, D acts on everything in the diagram in a way making it equivariant (in particular, acting on itself by conjugation). but this means that the action is trivial on K-groups, and what we obtain is:

$$\begin{tikzcd}
K_j(D, \mathbb{Z}_p) \arrow[r, "f^*"] & H^0(G, K_j(D \otimes_K L, \mathbb{Z}_p)) \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\simeq"] \\
K_j(K, \mathbb{Z}_p) \arrow[r, "f^*"'] & H^0(G, K_j(L, \mathbb{Z}_p)) \arrow[u, shift right, "\mathrm{Ird}"']
\end{tikzcd}$$

but unfortunately f^* are not isomorphisms, so we can't compare the K-theory of D to K in this way. But what if we replace everything by the valuation rings?

$$\begin{tikzcd}
\mathrm{Perf}\,\mathcal{O}_D \arrow[r, "f^*"] & \mathrm{Perf}(\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L) \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\not\simeq"] \\
\mathrm{Perf}\,\mathcal{O}_K \arrow[r, "f^*"'] & \mathrm{Perf}\,\mathcal{O}_L \arrow[u, shift right, "\mathrm{Ird}"']
\end{tikzcd}$$

we have a similar picture but we no longer have an adjoint equivalence, just an adjunction. But if we take THH with $\mathbb{Z}_p$ coefficients, again with the D-action,

$$\begin{tikzcd}
THH_j(\mathcal{O}_D, \mathbb{Z}_p) \arrow[r, "f^*", "\cong"'] & H^0(G, THH_j(\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L, \mathbb{Z}_p)) \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\not\simeq"] \\
THH_j(\mathcal{O}_K, \mathbb{Z}_p) \arrow[r, "\cong", "f^*"'] & H^0(G, THH_j(\mathcal{O}_L, \mathbb{Z}_p)) \arrow[u, shift right, "\mathrm{Ird}"']
\end{tikzcd}$$

we get horizontal but not vertical isomorphisms. We need to combine two approaches.

Theorem 2. Although the $\mathcal{O}_L$-order $\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L \subset D \otimes_K L$ is not maximal, it is still regular. Thus we get localization sequences in THH and K (where implicitly below we use dévissage) and traces between them:

$$\begin{tikzcd}
K(k_L \otimes_{k_K} k_L) \arrow[r] \arrow[d, "\mathrm{tr}"'] & K(\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L) \arrow[r] \arrow[d, "\mathrm{tr}"'] & K(D \otimes_K L) \arrow[d, "\mathrm{tr}"] \\
THH(k_L \otimes_{k_K} k_L) \arrow[r] & THH(\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L) \arrow[r] & \star
\end{tikzcd}$$

where $\star$ is not $THH(D \otimes_K L)$ because that doesn't actually fit. Instead, we name what belongs there $THH(\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L \mid D \otimes_K L)$.

Remark 3. This new flavour ofTHH group is a bit ad hoc, which is why the next statement is not at the level of spectra:

Theorem 4. There is a canonical isomorphism Nrd: $K_j(D, \mathbb{Z}_p) \xrightarrow{\cong} K_j(K, \mathbb{Z}_p)$ provided $j \geqslant 1$ given by the reduced norm. Moreover, $d \cdot \mathrm{Nrd} = N$ the usual norm, where we might need $p > 2$ in some cases.

Remark 5. The ℓ-adic case was settled 30 years ago by Suslin-Yufryakov.

If we modify our THH diagram using this idea, we get

$$\begin{tikzcd}
THH_*(\mathcal{O}_D \mid D, \mathbb{Z}_p) \arrow[r, "f^*", "\cong"'] & H^0(G, THH_*(\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_L \mid D \otimes_K L, \mathbb{Z}_p)) \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\simeq"] \\
THH_*(\mathcal{O}_K \mid K, \mathbb{Z}_p) \arrow[r, "\cong", "f^*"'] & H^0(G, THH_*(\mathcal{O}_L \mid L, \mathbb{Z}_p)) \arrow[u, shift right, "\mathrm{Ird}"']
\end{tikzcd}$$

and everything in sight is a graded module over $THH_*(\mathcal{O}_K \mid K, \mathbb{Z}_p)$. $THH_*(\mathcal{O}_D \mid D, \mathbb{Z}_p)$ is free of rank 1 over that, but we can't see this fact on K-theory and that's why the above theorem doesn't hold for K_0.

But we can define a Trd, Ird adjunction on $THH_*(- \mid -)$ by using the horizontal isomorphisms.

Remark 6. The same thing works for TC^- and TP.

Let's follow some equalities now:

$$\begin{aligned}
TC_0(\mathcal{O}_D \mid D, \mathbb{Z}_p) &= \pi_0 TC(\mathcal{O}_D \mid D, \mathbb{Z}_p) \\
&= \pi_0 \operatorname{Map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, THH(\mathcal{O}_D \mid D, \mathbb{Z}_p)) \\
&= \pi_0 \operatorname{Map}_{THH(\mathcal{O}_K|K,\mathbb{Z}_p)\text{-modules in CycSp}}(THH(\mathcal{O}_K \mid K, \mathbb{Z}_p), THH(\mathcal{O}_D \mid D, \mathbb{Z}_p))
\end{aligned}$$

and there's a similar result for TC^-:

$$TC_0^-(\mathcal{O}_D \mid D, \mathbb{Z}_p) = \pi_0 \operatorname{Map}_{THH(\mathcal{O}_K|K,\mathbb{Z}_p)\text{-modules in } \mathrm{Sp}^{B\mathbb{T}}}(THH(\mathcal{O}_K \mid K, \mathbb{Z}_p), THH(\mathcal{O}_D \mid D, \mathbb{Z}_p))$$

So let's pick a $y \in TC_0^-(\mathcal{O}_D \mid D, \mathbb{Z}_p)$ which is in the image of $1 \in TC_0^-(\mathcal{O}_K \mid K, \mathbb{Z}_p)$ which may be identified with $W(k_K)$ the Witt vectors over the residue field. We then obtain

$$\begin{tikzcd}
0 \arrow[r] & TP_1(\mathcal{O}_D \mid D, \mathbb{Z}_p)_\phi \arrow[r] \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\simeq"] & TC_0(\mathcal{O}_D \mid D, \mathbb{Z}_p) \arrow[r] \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\simeq"] & TC_0^-(\mathcal{O}_D \mid D, \mathbb{Z}_p)^\phi \arrow[r] \arrow[d, shift right, "\mathrm{Trd}"'] \arrow[d, phantom, "\simeq"] & 0 \\
0 \arrow[r] & TP_1(\mathcal{O}_K \mid K, \mathbb{Z}_p)_\phi \arrow[r] \arrow[u, shift right, "\mathrm{Ird}"'] & TC_0(\mathcal{O}_K \mid K, \mathbb{Z}_p) \arrow[r] \arrow[u, shift right, "\mathrm{Ird}"'] & TC_0^-(\mathcal{O}_K \mid K, \mathbb{Z}_p)^\phi \arrow[r] \arrow[u, shift right, "\mathrm{Ird}"'] & 0
\end{tikzcd}$$

where we get this middle adjoint equivalence whenever we pick a lift $\widetilde{y}$ of y. This also makes TC_0 free of rank 2 over $\mathbb{Z}_p$ (which does imply that there are choices). This implies the theorem now because the trace from $K_j \to THH_j$ is an isomorphism for D and K with $\mathbb{Z}_p$ coefficients when $j \geqslant 1$.

Final note: in work with Madsen, if we look at the map

$$TC_j(\mathcal{O}_K \mid K) \xrightarrow{\phi - \mathrm{can}} TP_j(\mathcal{O}_K \mid K)$$

we can tell it's an isomorphism for $j = 2k > 0$, and this points to the issue we have for $p = 2$.