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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Laurent Fargues

Talk Title: Geometrization of the local Langlands correspondence

Date: 3 / 28 / 19 **Time:** 2 : 00 am / [✓ pm] (circle one)

Please summarize the lecture in 5 or fewer sentences:

Bung_G is studied in many contexts, but in the p-adic local Langlands program it contains both arithmetic and geometric data.

Work with Scholze (based on earlier with with Fontaine) on this subject is detailed

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GEOMETRIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE

LAURENT FARGUES

Bun_G – what is it? Well, it's the moduli space of G bundles on a space X. Examples:

- A compact Lie group acting on a Riemann surface (Atiyah-Bott)
- A reductive group over $\mathbb{F}_q$ acting on a proper smooth algebraic curve over $\mathbb{F}_q$ (Geometric Langlands)
- Today: a reductive group over $\mathbb{Q}_p$ acting on "the curve" (Fargues-Fontaine)

It's a stack of an arithmetic/geometric nature. Arithmetically, Frobenius is incorporated into the geometry, which is unique among the above examples. For geometric, consider X a curve. If we take $\mathcal{F} \in \mathrm{Perv}(\mathrm{Bun}_G)$ a Weil sheaf, the trace of Frobenius gives an automorphic form on $G(F)\backslash G(A)$, where $F = \mathbb{F}_q(X)$.

Where's it from?

- At the ∞ place, Schmid: embeds Harish-Chandra discrete series into H_{L^2}(symmetric spaces)
- At the p place (here and after), Harris-Taylor: local Langlands for GL_n embeds into H(Lubin-Tate)
- Fargues: local p-adic Langlands for GL_n embeds into H(Rapoport-Zink spaces).
- F-F: fundamental curve of p-adic Hodge theory
- Scholze: perfectoid spaces, diamonds, local Shtuka,...

Bun_G encapsulates it all!

The Curve: a p-adic Riemann surface. Let $F/\overline{\mathbb{F}}_p$ be a perfectoid field. Consider $X_F \to \operatorname{Spec} \mathbb{Q}_p$ (uniformized), let ϕ be the Frobenius of F. Then $X_F = Y_F/\phi^{\mathbb{Z}}$ where $Y = \{0 < |p| < 1\}$ is built out of $W(\mathcal{O}_F)$ the Witt vectors with coefficients in $\mathcal{O}_F$.

Notes by Ian Coley.

Pick any perfectoid space $S/\overline{\mathbb{F}}_p$, which gives us a generalized $X_S \to \operatorname{Spec}\mathbb{Q}_p$. X_S we should think of as a family of $X_{k(s)}$ for $s \in S$.

Definition 1. $\mathrm{Bun}_\mathrm{G} \to \operatorname{Spec}\overline{\mathbb{F}}_p$ for G a reductive group over $\mathbb{Q}_p$: for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$ the fibre of the map should be G-bundles on X_S.

Theorem 2. It's a stack for the v-topology of Scholze (analogous to fpqc).

Structure:

Consider $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{\mathrm{unr}}}$ with an action σ of Frobenius. Let $B(G) = G(\breve{\mathbb{Q}}_p)/\sigma$-conjugacy, $b \sim gbg^{-\sigma}$.

Theorem 3 (F.)**.** $B(G) \to |\,\mathrm{Bun}_\mathrm{G}\,|$ is an equivalence where $[b] \mapsto [\xi_b]$ something wasn't described. But ξ_1 is the trivial G-bundle.

If b is "basic" (i.e. isoclinic), then ξ_b is semistable. So in this case, let J_b be the σ-centralizer of b, which is an inner form of G. All inner forms of G looks like this when $Z(G)$ is connected (e.g. GL_n). This is an extended *pure inner form* (Vogan).

Theorem 4 (F-Scholze)**.** $c_1 \colon \pi_0 \mathrm{Bun}_\mathrm{G} \xrightarrow{\sim} \pi_1(G)_\Gamma$, where $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Moreover, each connected component has a unique semistable point.

Interpreted, this means that $B(G)_{\mathrm{basic}} \cong \pi_1(G)_\Gamma$. Now let us focus on the semistable locus $\mathrm{Bun}_\mathrm{G}{}^{\mathrm{ss}}$ an open substack of Bun_G.

$\mathrm{Bun}_\mathrm{G}{}^{\mathrm{ss}} = \coprod_{[b]\ \mathrm{basic}}[\bullet = \mathrm{Spec}(\overline{\mathbb{F}}_p)/\underline{J_b(\mathbb{Q}_p)}]$ the classifying stack of proétale $J_b(\mathbb{Q}_p)$-torsors, and the underline means we restrict to locally profinite.

Geometrization: Let $\ell \neq p$. Let Π be an irreducible smooth $\overline{\mathbb{Q}}_\ell$-representation of $J_b(\mathbb{Q}_p)$. In general such Π are infinite dimensional, and is the local component at p of an automorphic representation of a global object. Π gives us $\mathcal{F}_\Pi$ a local system on $j\colon [\bullet/\underline{J_b(\mathbb{Q}_p)}] \to \mathrm{Bun}_\mathrm{G}$ a "purely stupid construction" and an open inclusion. Thus Π yields an ℓ-adic sheaf $j_!\mathcal{F}_\Pi$ on Bun_G (where $j_!$ is extension by zero).

Theorem 5 (F-Scholze)**.** Bun_G is ℓ-cohomologically smooth of dimension zero. The dualizing complex $K_{\mathrm{Bun}_\mathrm{G}}$ is isomorphic to $\overline{\mathbb{Q}}_\ell$ the trivial local system.

To prove this, we need to think on Bun_G through some concrete cohomologically smooth charts: developed to do this were some new kinds of algebraic geometry in Banach-Colmez spaces $= H^0(\text{curve, v.b.})$ an analogue of affine space.

Theorem 6 (F-S)**.** There's a Jacobian criterion of cohomological smoothness.

Example 7. $[\mathbb{B}^+_{\text{crys}}(-)]^{\phi^h=p^d}$ for $d, h \in \mathbb{N}^+$ is a Banach-Colmez space.

Hecke Correspondences: Let $S \in \text{Perf}_{\overline{\mathbb{F}}_p}$ be a perfectoid space, $S^\sharp$ an untilt of S over $\mathbb{Q}_p$. We can see $S^\sharp$ embeds in X_S as a Cartier divisor of degree 1. Then the formal completion of X_S along $S^\sharp$ correspond got Fontaine's $\mathbb{B}^+_{\text{dR}}$.

Let Div^1 be the moduli space of degree 1 Cartier divisors $= \text{Spa}(\mathbb{Q}_p)^\diamond/\phi^{\mathbb{Z}}$ a sheaf of untilts. For any finite set I, we get a span $\text{Bun}_{\text{G}} \leftarrow \text{Hecke} \xrightarrow{\star} \text{Bun}_{\text{G}} \times (\text{Div}^1)^I$ along with geometric Satake on the $\mathbb{B}_{\text{dR}}$-affine Grassmannian.

For any $\rho \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}(G^I)$ we get a kernel on Hecke_I. Because the map $\star$ is a locally trivial fibration in the $\mathbb{B}_{\text{dR}}$ affine Grassmannian, all this together gives us a cohomological Hecke correspondence on Bun_{G} (pullback, twist by kernel, pushfoward).

Using V. Lafforgue's machinery,

Theorem 8 (F-S)**.** If Π is a smooth irreducible representation of $G(\mathbb{Q}_p)$, we get $\phi_\Pi \colon W_{\mathbb{Q}_p} \to {}^LG$ the semisimplified Langlands parameter, giving a local Langlands correspondence for any group G. Moreover, is compatible with the cohomology of local Shtuka moduli spaces.

Proof. $\Pi \mapsto j_! \mathcal{F}_\Pi \mapsto$ apply the Hecke correspondence and iterate which gets you to the parameter. ☐

There should also be a way to get from ϕ to $\mathcal{F}_\phi \in \text{Perv}_{\overline{\mathbb{Q}}_\ell}(\text{Bun}_{\text{G}})$. Very little is known, but we do know GL_1:

Theorem 9 (F)**.** For all $d \geqslant 3$, $AJ^d \colon \text{Div}^d \to \text{Pic}^d \subset \text{Bun}_{\text{GL}_1} = \text{Pic}$ which sends $D \mapsto \mathcal{O}(D)$ is a proétale locally trivial fibration in simply connected diamonds, where $\text{Div}^d = (\text{Div}^1)^d/\sigma_d$