Fall 2001

Joe P. Buhler (jpb@msri.org)
Elwyn Berlekamp (berlek@math.berkeley.edu)

Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. In a pitch-black room you are handed a deck of n cards, and are told that 10 of them are face up in the deck and the others are face down. You can manipulate the cards (say, to count them and determine n) but you cannot see the cards at all.

For which values of n can you separate the deck into two piles each of which is guaranteed to have the same number of face-up cards?

Remark: This is an old puzzle that recently received a burst of popularity on the internet after it was given as the weekly puzzle on the "Car Talk" NPR radio show.

Discussion: For $n > 10$, put the deck into piles of size 10 and $n - 10$, and turn the pile of size 10 upside down.

Problem 2. A village has 24 voters, half Tories and half Whigs. All of them faithfully vote for their party's candidates. A court requires the village to be partitioned into 8 voting districts, each of which must consist of exactly three voters. Each district will then elect one representative to the village council. The details of the assignments of the 24 voters to the 8 districts are relegated to the current mayor of the village.

A naïve observer predicts that, since the two parties have equal numbers of voters, neither will be able to obtain a majority on the village council.

(a) Explain why the naïve observer is wrong.

(b) Make your own prediction: what fraction of the seats will be carried by the mayor's party?

Discussion: a. The naive observer is wrong because he doesn't understand gerrymandering, as we'll now show.

b. The mayor will stuff 3 voters of the opposite party into each of two selected "unfriendly" districts, and then distribute the other 6 opposing voters evenly among the remaining "friendly" 6 districts. All twelve of his own supporters will be distributed, two each, among the friendly districts. Then the mayor's party will have a majority in all six of the friendly districts, and dominate the village council, 6 votes to 2.

Problem 3. A team of three plays the following game. After an initial session, during which they confer about what strategies they will follow, the teammates are put into separate hotel rooms. A judge flips a fair coin for each player and assigns that player a bit (0 or 1). Each player is then given the list of bits assigned to other players (matched to their names); however, players are not told what they themselves got. No communication is possible between the hotel rooms.

A. At this point, each player must send one of the following three statements to the judge: "my bit is 0", "my bit is 1", or "I pass". If at least one player doesn't pass and all non-pass statements are true, the team collectively shares a million-dollar prize. If all players pass or someone makes a false statement, they get nothing.

   An obvious strategy gives the team a 50/50 chance to win: they agree that they will all pass except for one specific player, who will guess. Can you devise a strategy that gives the team a better chance of winning?

B. Image the same setup, but players are not allowed to pass: each must say “my bit is 0” or “my bit is 1”. The team shares a million-dollar prize if and only if a majority of the statements are true. Find a good strategy for the team.

C. Same two puzzles as above for 7 people. Or for an arbitrary number (except that B requires that there be an odd number of people).

Discussion:
A. Amazingly, the following strategy gives the team a 75% chance of winning (as is easily verified): if you see opposite bits (colors) then pass; if your two teammates’ bits are the same, then assert that your bit will be the reverse.

When any individual player fails to pass, he will be wrong 50% of the time. The above strategy coordinates the teammates in the sense that if they are wrong, then all three of them will be wrong. Geometrically, we can imagine the state to be a vertex of a cube (corresponding to the 8 possible bit states). An individual player knows that the outcome is one of two possible vertices The 2-element set $C := \{(0, 0, 0), (1, 1, 1)\}$ is a “covering code” in the sense that every state can be obtained from an element of C by changing (at most) one bit. The team then behaves as if the outcome is **not** in C; if they are right, then exactly one player won’t pass, and that player will make a correct statement. The chance that they are in C is of course $2/8 = 1/4$, so their chance of success is $3/4$.

Note that we could view the team’s strategy as assigning a “pass” or a directed arrow to each edge of the cube, since each player “is on” an edge of the cube.
B. Retaining the geometric interpretation from above, we see that we must assign an arrow to every edge of the graph, since passing is no longer allowed. We retain the choice above — each edge connected to C is oriented away from C. We call these votes (edges) **decisive** and oriented the remaining edges so that each vertex not in C has two incoming edges (including the decisive one) and one outgoing edge. One easy way to do that is to remove all 6 decisive edges and observe that the remaining 6 edges form a cycle which can by cyclically oriented to achieve our goal.

Thus the team still has a 75% of winning under the majority rules.
C. As might be guessed (and isn’t too hard to prove), the solution to the case of n people is equivalent to finding a subset C of the vertices of the unit n-cube with the property that every vertex is within a distance one of C. This is called a “covering code” and is a complementary notion to usual error-correcting codes. For $n = 2^k - 1$ Hamming codes are perfect, and are therefore also covering codes; they give the team a probability of $n/(n + 1)$ of winning (so that, curiously, the probability of winning goes to 1 as n goes to ∞). For arbitrary n, the situation is more complicated, and best possible values are known only for small values of n. Many details, references, and variants can be found in “Hat Tricks” in the mathematical entertainments column of the Fall 2002 issue of the *Mathematical Intelligencer*.