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Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. (Matthew Hubbard) 101 ants start out randomly placed on a circle of circumference 1. At time 0, they begin moving in a random direction at 1 meter per minute, bouncing off each other instantaneously. What is the probability that after 1 minute Alice will be exactly back to her starting place?

Discussion: Using the baton idea from the previous two ant puzzles, all batons are exactly back at their starting position after 1 minute. The relative ordering of the ants remains unchanged, so the problem is to figure out how many positions the ants have rotated relative to their batons.

Suppose k batons move clockwise, and $j = n - k$ batons move counterclockwise. Then each of the k batons passes each of the j batons exactly twice. The total affect of each of the k batons is to rotate the ants relative to the batons two places in one direction, modulo n. So the total affect of all batons is to rotate them $2k$ places, (or equivalently, $-2j$ units) modulo n. It follows that all ants end up back with their original batons, and hence, exactly where they started, if and only if $2k \equiv 0 \mod n$. In all other cases, every ant ends up in a different place than she started. The congruence $2k \equiv 0 \mod n$ can be satisfied in exactly three ways: $2k = 0$, n, or $2n$. If n is 101, or any other odd number, then $2k$ cannot be n, and so the ants can end up exactly where they started iff they all face the same direction, which happens with probability 2^{1-n}. If n is even, there is the additional chance that exactly half of them face each way, which happens with probability

$$2^{1-n}\binom{n}{n/2}.$$

For large even n, the probability of them all facing the same way is insignificant compared to the probability of half of them facing each direction, and by Stirling's formula, this is very close to $\sqrt{2/(\pi n)}$.

Problem 2.

```
  MAAAMS
+   SIAM
+MEETING

=
      IN
+ATLANTA
+     01
+     06
+   2005
```

Discussion: A solution is an appropriate one-to-one mapping of the capital letters into the ten decimal digits. Since leading zeroes are traditionally omitted, a solution would not be considered appropriate if the value 0 were assigned to any of the letters `M`, `S`, `I`, or `A`.

Some people who are numerically inclined get more enjoyment from puzzles like this than others do from crossword puzzles.

Each of the seven columns, corresponding to the 10^n th digits for n equal to 0 through 6, can give us an equation among the seven unknown digits and the unknown values of certain carries. But we know that each of the 9 capital letters must be a distinct integer from 0 to 9. So we may view a cryptogram such as this as a one-dimensional jig-saw puzzle, in which we need to fit the pieces M, E, T, I, N, G, A, S, L into the sequence of spaces corresponding to the digits 0123456789. In the present puzzle, there are only 9 letters, so one space must be unused. While working on such a jig-saw puzzle, useful partial results often consist of inequalities, in which we can establish that the larger-valued piece must be placed to the right of the smaller-valued piece, or perhaps a forced fit among a sequence of pieces (possibly including blanks), in which we deduce the precise small difference between the values of certain pieces.

There are various heuristic procedures which strive to "guess" a solution and then verify that it works. The more challenging problem is to find ALL solutions, methodically, without use of any computing devices. We now give such a solution.

We begin by showing the top three carries explicitly:

```
 MEE
+ MA
----
 abc
----

 ATL
```

Here each of a, b, c, is either zero or 1. The leftmost column (the millions column in the original statement of the problem) requires that $M + a = A$. As the carry-out from the ten-thousands' column, a must be either 0 or 1, and since M cannot equal A, we must have $a = 1$ and $A = M + 1$. This gives us a "forced fit" placing the A piece of our jigsaw puzzle immediately to the right of the M piece. So we join this pair of pieces into the entity MA. The hundred-thousands and ten-thousands columns next give these equations:

$$E + M + b = 10 + TE + A + c = 10b + L$$

and since $A = M + 1$, it follows that $L + 10(b - 1) = 1 + c - b + T$ This has no solutions with $b = 0$, so $b = 1$, and it becomes $L = c + T$, hence $L > T$. The thousands column precludes $c \geq 2$, so $c = 1$, $L = T + 1$, and we have another forced fit of the T and L pieces into the entity TL. The hundred-thousands and ten-thousands columns now reduce to the same equation, namely

$$E + M = 9 + T,$$

which implies that $M > T$, and $E > T$, and hence that our E and MA pieces must both lie to the right of the TL piece.

After replacing the M in the ones column by its equivalent value $A - 1$, and then removing the canceling A's above and below the line in the ones column and in the thousands column, we reduce the original cryptogram to the following:

```
 MEETING
+ MA0AMS
+   SIA0
--------
 111def0
--------

 ATL2NT0
+     IN
+     13
--------
    ghi0
--------
```

This thousands columns now gives

$$\mathtt{T} + \mathtt{S} + \mathtt{d}' = 12,$$

where $\mathtt{d}' = \mathtt{d} - \mathtt{g}$. Since d is 0 or 1 or 2, and g is 0 or 1, the possible values of d' are -1, 0, 1, and 2.

We are now ready to tabulate all solutions of the leftmost four columns. We first select a value of S, then a value of T consistent with the equation $\mathtt{T} = 12 - \mathtt{S} - \mathtt{d}'$, then $\mathtt{L} = \mathtt{T} + 1$, then values of M, $\mathtt{A} = \mathtt{M} + 1$, and E according to $\mathtt{M} + \mathtt{E} = \mathtt{T} + 9$. Since we know T to be smaller than L, MA, and E, we expect S to be large so we itemize the values of S in top-down order:

With S = 9:

```
0123456789  d'
 TLMA  E S  2
 TL MAE  S  2
 TL E MA S  2
 TLE   MAS  2
  TLMA E S  1
  TL EMA S  1
  TLE  MAS  1
   TLMAE S  0
   TLE MAS  0
    TLEMAS -1  ***
```

With S = 8,

```
0123456789  d'
  TLMA ES   2
  TL EMAS   2
   TLMAES   1
```

Evidently, there are no more solutions, because $\mathtt{S} \leq 7$ forces $\mathtt{T} \geq 3$, and this forces some collision(s).

The row marked ***, with $S = 9$, $T = 4$ requires $d' = d - g = -1$, which requires $N = 9$. The collision $N = S$ invalidates this row.

We still have three remaining equations, corresponding to the ones, tens, and hundreds columns. Each of them contains two of the three remaining unknowns, I, N, and G. We begin with the ones column, because it is least impacted by the unknown carries. Our plan is to use the constraint of the ones column to fit the N and G pieces into our puzzle, and then use the hundreds column to determine I. This will allow us to evade the precise values of e,f,h, and i until the final check.

The ones column requires that, modulo 10, $S + G \equiv N + 3$. So if $S = 9$, $G \equiv N + 4$, and we can insert all possible fits of the N and G pieces into our table of candidate solutions as follows:

```
0123456789  d'
GTLMA NE S  2
 TLNMAEG S  2
 TL E MA S  2  ***
NTLEG  MAS  2
GTLE  NMAS  2
G TLMANE S  1
N TLGEMA S  1
 NTLEG MAS  1
G TLE NMAS  1
  TLNEMAGS  1
  GTLMAENS  0
G  TLENMAS  0
  NTLEGMAS  0
```

Likewise, if $S = 8$, then $G \equiv N \bmod 5$. Since any fit of G and N gives another fit with G and N swapped, we now represent both G and N by the same the letter U

```
0123456789  d'
 UTLMAUES   2
  TLUEMASU  2
   TLMAES   1  ***
```

Since G and N can't fit into either of the rows marked ***, these rows can be eliminated.

At this point, we find it helpful to eliminate the discrepancy between d' and $d = d' + g$. If g is nonzero, then we must have $N = 9$, and the only row in which that can occur is with $S = 8$, $A = 7$. But in this row the only possible values of I are 0 and 1, whence d, the carryout from the hundreds column cannot exceed 1, thus failing to meet that row's requirement that $d' = 2$. So we can henceforth assume that any solution must have $d' = d$ and that $N < 9$.

The hundreds column gives this equation, $2I = 10d + N - A + h - e$, or $I = (10d + N - A)/2 + (h - e)/2$.

In each row, the first term is known, and the second term is confined to a small range: -1/2, 0, 1/2, or 1. So for each row, we tabulate the possible values of I:

```
0123456789  d   I
GTLMA NE S  2  11,12  ***
 TLNMAEG S  2   9,10  ***
```

```
NTLEG  MAS  2   6, 7
GTLE  NMAS  2   9,10  ***
G TLMANE S  1   5, 6  ***
N TLGEMA S  1   1, 2
 NTLEG MAS  1   1, 2  ***
G TLE NMAS  1   4, 5  ***
  TLNEMAGS  1   3, 4  ***
  GTLMAENS  0   1, 2
G  TLENMAS  0  -1, 0  ***
  NTLEGMAS  0  -3,-2  ***

 GTLMANES   2  10,11  ***
 NTLMAGES   2   8, 9
  TLNEMASG  2   8, 9  ***
```

The rows into which neither of the possible values of `I` can be fit have been marked with `***`. From the others, we narrow our choices to the four following final candidates:

```
0123456789  d
NTLEG IMAS  2
NITLGEMA S  1
 IGTLMAENS  0
 NTLMAGESI  2
```

Finally, each of these four remaining candidates needs to be examined to see whether it meets the unused constraint of the tens column (including values of the `f`, `i`, `e`, and `h`, which we have thus far evaded). Three of them flunk that test, leaving this unique solution

```
0123456789
NITLGEMA S
```

Problem 3. A set A of positive integers is said to have density d if the fraction of integers in $[1, n]$ that are in A approaches d as n goes to infinity, i.e.,

$$d = \lim_{n\to\infty} A(n)/n$$

where $A(n)$ is the cardinality of the set of elements in A of size at most n.

Find two sets of density one-half whose intersection does not have a density, i.e., the relevant limit above does not exist.

Problem 4. For positive real x less than 1, define

$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \cdots$$

Does $f(x)$ have a limit as x approaches 1 from below? If so, what is the limit?

Discussion:

Surprisingly, the limit does not exist. This result first appeared in a paper by Hardy, "On certain oscillating series", *Quarterly J. Math.* **38** (1907), 269–288 (in the sixth volume of Hardy's collected papers, pp. 146–168), where he comments that no completely elementary proof seems to be known. Noam Elkies posed the problem to us — see abel.math.harvard.edu/∼elkies/Misc/sol8.html; Tanguy Rivoal brought the reference to Hardy to Elkies' attention. In addition, a number of people wrote us with interesting comments or elaborations on the problem, including Herb Wilf, Neil Calkin, Jon Keating, William Kahan, and others. In particular, Poisson summation can be used to give a very precise understanding of the behavior of $f(x)$ as x approaches 1; e.g., see "Summability of gap series," J. P. Keating and J. B. Reade, *Proc. Edin. Math. Soc.*, **43** (2000), 95–101. In fact, as x goes to 1, the function $f(x)$ oscillates around $1/2$ with an amplitude of approximately 0.00275; an interesting graph to this end can be found on Elkies' web site. In due course, our web page will contain a discussion of some of these, and other, ideas on this problem.

The easiest proof of nonconvergence that we know can be found on Elkies' web page. Since f satisfies the functional equation

$$f(x) = x - x^2 + f(x^4),$$

just finding *one single* $u < 1$ for which $f(u) > 0.5$ shows that the desired limit does not exist, because the values of f at u, $u^{1/4}$, $u^{1/16}$, $u^{1/64}$, $\ldots$ keep increasing (note that $x - x^2 > 0$). Now, modern numerical engines (or even tedious hand computation) can verify $f(0.995) = 0.50088\ldots > 0.5$. Hardy himself gave similar elementary arguments to show that $x - x^a + x^{a^2} - x^{a^3} + x^{a^4} \pm \cdots$, for integer $a > 2$, has no limit as x approaches 1 from below; his assertion that no such elementary argument seemed feasible for $a = 2$ might have been due to an aversion to calculation.