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Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. Prove that if two medians in a triangle are equal in length, the triangle is isosceles.

Discussion: One way to use this is to use the well-known fact that if medians AD and BE in a triangle ABC intersect at O, then

$$AO : OE = BO : OF = 2.$$

It follows that the triangles AOF and BOE are congruent, since the angles $\langle AOF$ and $\langle BOE$ are equal, and the sides meeting at O are the same length (since the medians have the same length). Thus $AF = BE$ and $AC = 2 \cdot AF = 2 \cdot BE = BC$ so that the triangle is isosceles as claimed.

It is also possible to prove this in various ways using analytic geometry. The BAMO webiste at mathcircle.berkeley.edu has answers to this year's exam and, in particular, has 4 different answers to this question.

Problem 2. There are 1000 cities in the country of Euleria, and some pairs of them are linked by dirt roads. It is possible to get from any city to any other city by traveling along these dirt roads. Prove that the government of Euleria may pave these roads so that every city will have an odd number of paved roads leading out of it.

Discussion: Take any paving (e.g., the empty paving). The total number of cities with an odd number of incident paved roads is even. (The total number of pairs (c, r) where c is a city and r is an incident paved road is even, since each road occurs in exactly two ordered pairs; thus the number of cities c with an odd number of incident r must be even.)

If there are no cities with an even number of incident paved roads we are done. Otherwise, choose two cities c, c' with an odd number of incident paved roads. Choose a path between them, and reverse the paving/unpaving on each one. The cities c and c' now have an even number of incident roads, and no other "parity" of a city is changed. Continuing in this way gives a paving as desired.

See the BAMO website mentioned above for other proofs.

Problem 3. (a) Place four integers at the corners of a square. At the center of each edge, write the absolute value of the difference between the integers. In algebraic language, the cycle (a, b, c, d) leads to the cycle $(|a - b|, |b - c|, |c - d|, |d - a|)$. Repeat this operation. Prove or disprove: any starting sequence of integers leads to the all-zero square in finitely many iterations.

(b) Same problem with arbitrary positive real numbers allowed as starting values.

Problem 4. Prove or disprove: For all positive integers n,

$$\left\lceil \frac{2}{2^{1/n} - 1} \right\rceil = \left\lfloor \frac{2n}{\log 2} \right\rfloor .$$