Spring/Summer 2011

EMISSARY

Mathematical Sciences Research Institute

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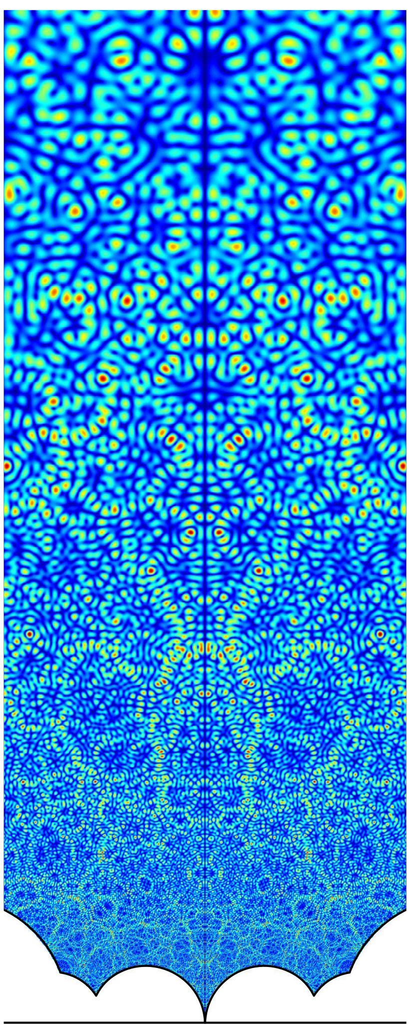
An Eventful Spring

A word from Director Robert Bryant

The spring semester here at MSRI has been an especially active one. In addition to our two scientific programs, Arithmetic Statistics and Free Boundary Problems, which have attracted two enthusiastic groups of mathematicians to keep the building busy and which have had numerous well-attended workshops, we've had a variety of other exciting activities and workshops.

(continued on page 2)



Fredrik Strömberg

What does this stunningly beautiful pattern have to do with counting primes? See article starting on page 8.

Free Boundary Problems

Henrik Shahgholian

Many problems in physics, industry, finance, biology, and other areas can be described by partial differential equations that exhibit a priori unknown sets of positive codimension, such as phase boundaries, optimal stopping strategies, contact sets or shocks, as the case may be.

The study of such sets, known as free boundaries, has a central role in such problems. The geometry and analysis of the underlying mathematical problems are the qualitative features that have provided the greatest challenges for mathematicians for decades and still pose many open problems.

Paradigmatic examples are the classical Stefan problem and more general models of phase transitions, where the free boundary is the moving interface between phases. Other examples come from problems in surface science, plastic molding and glass rolling, filtration through porous media, where free boundaries occur as fronts between saturated and unsaturated regions, and others from reaction-diffusion, fluid dynamics, mathematical finance, biology, and water wave theory.

Many of these applications are divided into one-, two-, and multiphase problems, depending on the number of regions in which the underlying field equations are to be solved.

FBPs is an area of mathematics and its application that remains theoretically vibrant while giving unique practical insight, and it will surely stay on the mathematical map forever.

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A Word from the Director

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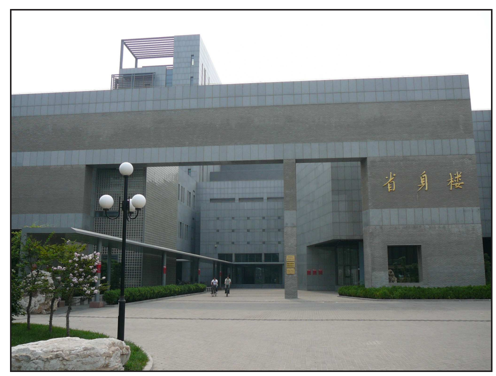
In February, we had the pleasure of hosting the Workshop on Mathematics Journals, which featured many speakers giving their perspectives on the challenges that lie ahead for mathematics journals. It was organized by James M. Crowley (SIAM), Susan Hezlet (London Mathematical Society), Robion C. Kirby (University of California, Berkeley), and Donald E. McClure (AMS), and the speakers included working mathematicians, librarians, publishers, and a number of other individuals, all of whom contributed to a lively discussion on this subject that will affect all of us who create and use the mathematics literature. While it would be too much to expect everyone to agree on what policies the mathematics community should adopt, it was encouraging to hear the many constructive approaches being taken. For those who are interested, videos of many of the talks are online at MSRI's VMath web site and the organizers have posted a white paper that summarizes the points of view that were represented on the conference web page (http://tinyurl.com/3qhwyuf).

We've also had quite a semester of theater and mathematics. In February, MSRI hosted the playwrights of PlayGround for a discussion of how mathematicians think about number theory and culture. The playwrights each then had nine days to write a ten-minute play using the theme "Kingdom of Number". The more than twenty plays submitted were then judged by a panel consisting of mathematicians and theatrical producers, and the six best were given staged readings at the Berkeley Repertory Theatre on February 21. It was a nearly sold-out evening, and the six plays that were performed were well-received, continuing the tradition that MSRI night at PlayGround is one of the most successful PlayGround events each year.

In scientific outreach, this spring MSRI became one of the founding members of the Mathematics of Planet Earth 2013 program, which hopes to focus attention on the mathematical challenges inherent in addressing the global problems of sustainability, managing diseases and epidemics, management of resources, and studies of climate and its effect on life on earth. This program is, appropriately enough, global in scope, and MSRI is pleased to be a part of this consortium of mathematics institutes worldwide who are addressing these problems. You can find out more about MPE2013 and MSRI's involvement by going to the web site www.mpe2013.org. We've also begun participating in the planning for the first Mathematical Congress of the Americas, which will take place in August 2013 at a yet-to-be-determined location in the Western Hemisphere.

Looking forward to this fall, we have been busy with preparations for the Chern Centennial Conference, which will celebrate the 100th anniversary of the birth of MSRI's founding director, Shiing-shen Chern, with a two-week conference on his mathematical legacy. The first week (October 23 to 28) will take place at the Chern Institute of Mathematics in Tianjin, and the second week (October 30 to November 5) will take place at MSRI. (Please visit our web site for more details, as they become available. It will be a memorable conference that you won't want to miss.)

Richard P. Stanley

Two views of the Chern Institute, MSRI's sister institution in Tianjin, China.

At the beginning of April, MSRI hosted the well-known monologuist Josh Kornbluth for two sold-out evenings during which he performed his monologue *The Mathematics of Change*, which tells the story of his stressful encounters with calculus during his freshman year at Princeton. It is a moving, thoughtful, humorous piece, and those who missed his live performances of this monologue will soon be able to see it on the concert DVD, since the performances at MSRI in the Simons Auditorium were filmed. (Thanks to our donors Jerry Fiddler and David Fuchs for their support of this project, which will give a national audience a chance to see this work, as well as a chance to see MSRI on film.)

Finally, at the end of April, MSRI and UC Berkeley hosted the play *MSI: The Anatomy of Integers and Permutations*, by the theater producer Jennifer Granville and her brother, the mathematician Andrew Granville. This highly original production, which teaches concepts in group theory and number theory through a murder mystery, received its West Coast premiere at MSRI, and perhaps we'll get a chance to see it filmed as well.

This has also been a semester of considerable change of personnel here at MSRI. We are pleased to be welcoming new members of the Board of Trustees: Georgia Benkart (University of Wisconsin-Madison), Neil Chriss (Hutchin Hill Capital), and José Scheinkman (Princeton University). At the Trustees meeting in March, the Board elected three new members of the Human Resources Advisory Committee, who will serve three-year terms: Minerva Cordero-Epperson (University of Texas at Arlington), William A. Hawkins, Jr. (Mathematical Association of America), and Duane Cooper (Morehouse College).

Finally, one more personnel change. Many of you have come to know our Development Director, Jim Sotiros, as a tireless, highly effective advocate for MSRI, and those of us who have worked with him know how dedicated he has been and how very much he has contributed to MSRI's growth during the more than 10 years. Sadly for us, Jim has decided to take up another challenge, development for a health-provider foundation that is near his home in Vallejo. We wish Jim the very best in his new endeavor!

Free Boundary Problems

(continued from page 1)

Historical development

Free boundary problems can be traced back to Neumann's work on the Stefan problem in the early 1800s. Later major advances in the nineteenth century concerned free surfaces flows (Helmholtz) and shock waves. Works of T. Levi-Civita (1925) and D. J. Struik (1926) on water waves also had a certain impact on the development of the theory.

In the early years of the theory, most problems concerned free boundaries related to jets and cavities, and the use of complex analysis and function theory were the main tools. Here one should mention names such as Birkhof and Zarantonello (early 1950s), and their 1957 book on the topic.

From the early 1950s until the early 1970s mathematicians including H. Lewy, G. Stampacchia, and Lions contributed to the growth of the field, and eventually the weak and variational formulation and techniques for these problems were developed. There are many names here unmentioned, but not forgotten.

Duvaut and Lions were probably two of the first to make an extensive list of problems available to mathematical community, in their book. At this stage of the development of the topic, several deep contributions were made to obtain regularity of solutions and to develop the theory towards other directions.

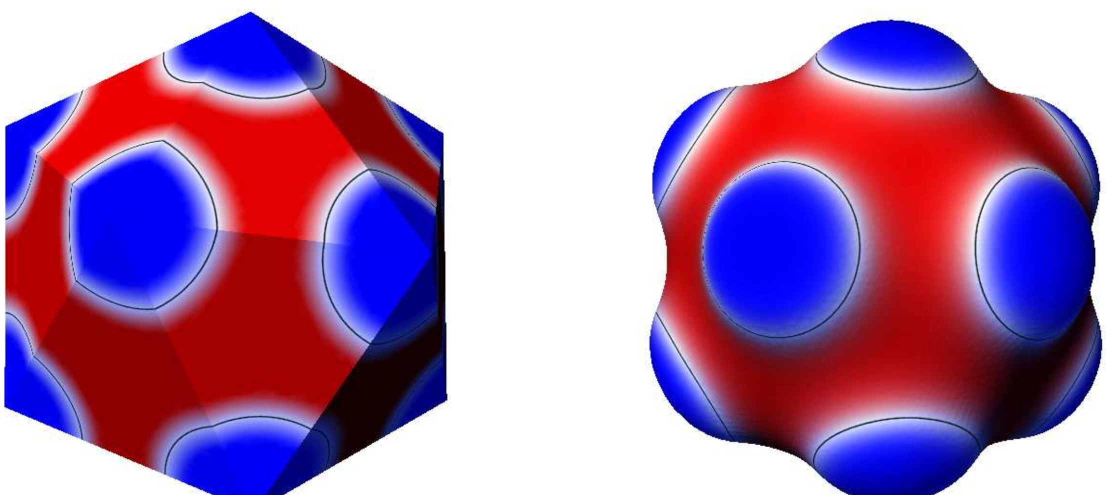
One of the earliest manifestations of FBPs as a separate mathematical discipline was the proceedings volume *Moving boundary problems in heat flow and diffusion* (Oxford, 1974). Subsequent conferences placing due emphasis on the key roles of mathematical and numerical analysis were held in Gatlinburg, Durham (UK), and Berlin. These events culminated in the famous event in Montecatini, which set the pattern for the series of which the MSRI program on FBP is the latest.

Recent theoretical developments

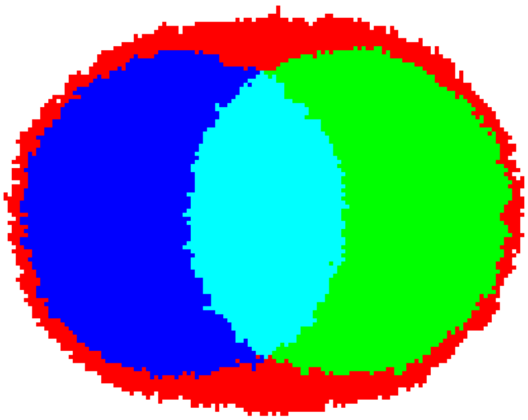
Nonlocal operators. Many recent developments involve phenomena where the diffusion process is of a nonlocal nature: in continuum mechanics and fluid dynamics, due to the presence of many scales, as in polymers, the existence of a global interaction through force fields, or in cases of surface diffusivity induced by spatial phenomena. Indeed, if heat flow is blocked by an insulated wall, heat will propagate along the wall by the influence of internal heat, through the Dirichlet to Neumann nonlocal diffusion kernel. This is the case of the quasigeostrophic equation or models of planar crack propagation in viscoelastic solids. Such behavior also takes place in probability, when the random kicks affecting the particles cease to be "infinitesimal" and become "jump processes" (Levi processes). Many phenomena involving phase transitions and free boundaries reappear in this context.

Nonlocal behavior has many aspects and scales (thin membrane, thick membrane, different scales for interior and boundary diffusivity, and on on) and presents new mathematical difficulties. It plays a crucial role in applications such as elasticity (the Signorini problem), optimal insulation and mathematical finance (American option with Levi process).

Two-phase and multiphase problems. Many phenomena modeled by PDE give rise to several phases of behavior; hence two-phase and multiphase free boundaries are natural in these problems. They may arise as limit cases of solutions to systems of equations. The occurrence of several phases in flow problems also occur in industrial applications, especially in the oil industry, where they can involve oil, gas, air, salt water, sand, etc. The well known Muskat problem is an example of such phenomena. Another problem is segregation in reaction-diffusion problems, which models competition between several species. The theory of these problems is still at an embryonic stage of development.



Initial configuration and equilibrium shape for a vesicle with multiple rafts. Spontaneous curvature together with different bending rigidities can stabilize multiple domains of one phase embedded in another phase on vesicles close to spheres. Obtained by gradient flow relaxation of an initial shape shown on the left. Such multiple raft equilibria arise when the material parameters in the two phases are different. Graphics by Charlie Elliott and Bjorn Stinner (Warwick).



Logarithmic fluctuations from circularity: two-source internal DLA cluster built from overlapping single-source clusters. Graphics by Lionel Levine.

Combinatorial aspects and random walks. Many applied problems result in particle interactions in various forms. Such problems, driven by exterior or interior forces, chemical reactions, or population growth, have been studied to some extent within applied sciences. When particles move (e.g., cell growth, crystallization, discrete flows) randomly and occupy regions, they interact with other particles in their environment. Such interaction can be of various forms: they can be annihilated, or annihilate other particles, they can freeze or vaporize and behave in many other ways. When there is a source present, these different particles occupy different regions, which will expand. Such movements usually are described by random walks, and the density representing the population, or the amount of chemical substances, are given by discrete harmonic functions.

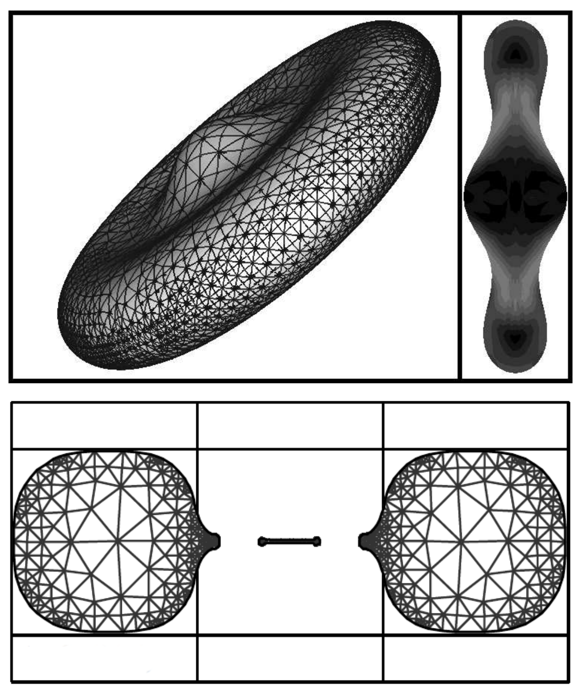
Models for one phase version of these discrete problems have recently been developed by several mathematicians. The problem has some history in combinatorial problems such as chip firing, random algebraic sums (smash-sum), and coloring problems.

The two- and multi-phase versions of such problems have yet not been developed, even though very many corresponding models have been proposed. As an example one may consider two non-mixing flows that vaporize or freeze at the interface between them. This results in an interface between two regions of occupied particles, which evolves as a curve in a plane or a surface in space. The corresponding particle dynamics involves random walks of two types of particles that occupy previously unoccupied space, and upon interaction they annihilate each other on the interface. There is an abundance of similar models with a variety of interactions, which need to be discovered and studied.

Shock waves. The study of solutions with shocks in various models of compressible fluid dynamics leads to well-known free boundary problems. One class of such problems involves steady and self-similar solutions of the multidimensional Euler equations. Shocks may correspond to discontinuities in the fluid velocity, as well as the density and the pressure, which mathematically are discontinuities in the solutions or in their derivatives depending on the model equations. Important physical applications involving shocks include self-similar shock reflection-diffraction and steady shocks in nozzles and around wedges/cones or airfoils. Such shocks are important in the mathematical theory of multidimensional conservation laws since steady/self-similar solutions with shocks are building blocks and asymptotic attractors of general solutions. In recent years, mathematical progress has been achieved in several long-standing problems, in particular in shock reflection-diffraction problems, for potential flow in which many different flow regimes are possible. Future challenges include:

- complicated cases of regular shock reflection-diffraction and other patterns for potential flow;
- extension of the results of shock reflection-diffraction, including regular and Mach patterns, to the full Euler equations;
- construction of steady solutions in a nozzle, called de Laval nozzle, or around a nonstraight wedge/cone or an airfoil; and
- analysis of free boundaries in solutions to the multidimensional Riemann problem.

Numerical developments. In the early days, numerical approaches to FBP were frequently ad hoc, based say on trial free boundary methods or coordinate transformations. Systematic approaches based on variational methods, variational inequalities and weak formulations of degenerate nonlinear parabolic equations were developed as the mathematical theory became available. In recent years important developments include the following:



A fluid-biomembrane (top) and a Hele–Shaw model for electrowetting on dielectric (bottom). Graphics by Ricardo Nochetto.

- diffuse interface models (such as phase field and Cahn–Hilliard models) with applications to curvature flows, solidification and phase transformations in material science;
- level set methods for evolving fronts including applications to fluid flow and image processing;
- variational front tracking methods for geometric PDEs, for instance interfaces involving curvature effects (such as surface tension and bending);
- extensive mathematical contributions to the stability, well posedness and rigorous error analysis of discrete approximations to free boundary problems and degenerate nonlinear elliptic and parabolic equations;
- adaptive methods appropriate for free boundary and interface problems.

The needs to simulate ever more complicated large systems, stemming from the increase in computing power and the development of computational tools, stimulates new questions and problems for analysts. Thus there is a natural symbiosis between analysts, modelers and computational mathematicians. Increased computing power together with the demand of applications have led to the study of systems of PDEs in domains with complex morphology. These complex multi-physics models frequently involve interfaces and free boundaries. Although one may obtain detailed information about sub-problems (say the obstacle problem) or local in time existence results using analytical techniques, in scientific and engineering applications, the full complex system is required to be simulated. Nonlinear degenerate PDEs with interfaces and free boundaries are notoriously difficult to solve numerically. The best numerical methods depend on good analytical approaches, and sometimes they promote new advances in PDE theory, such as the formulation of mean curvature flows beyond the onset of singularities. Future trends in numerics may include:

1. Surface finite elements: geometric PDEs, surface processes on interfaces.
2. Diffuse interface and phase field methods: two-phase flow with surfactants, phase transformations in materials.
3. Numerical optimization of free boundaries: control and inverse problems.
4. Burgeoning applications in biology and medicine.
5. Numerical methods for fully nonlinear equations.
6. Numerical methods in homogenization, random media and random surfaces including stochastic equations.
7. Computational methods for free boundary problems for shock waves and compressible flows.
8. Adaptivity (mesh refinement, coarsening and smoothing) for surfaces including topological change.

I wish to thank several colleagues for their contributions to this note: Luis Caffarelli, John Ockendon, Charlie Elliott, Ricardo Nochetto, and Gui-Qiang Chen.

Focus on the scientist: Arshak Petrosyan

Nicola Garofalo

Norayr Matevosyan

Arshak Petrosyan began his research in free boundary problems with his Ph.D. dissertation on preservation of log-convexity in Bernoulli type FBP for the heat operator and the existence and uniqueness of the solutions to these problems. His work has since considerably expanded, with particular focus on the understanding of the most difficult aspects of the theory, such as the regularity of the free boundary, the classification of the so-called blow-up solutions, and their uniqueness.

Most of his more recent work centers around the use of deep monotonicity formulas, an area in which he has become a leading expert. Significant examples include his 2004 paper with L. Caffarelli and H. Shahgholian in *JAMS*, which treats a parabolic Stefan-type FBP with no assumptions on the sign of the solution or of its time derivative; his 2009 article with N. Garofalo in *Inventiones* on the singular set of the free boundary in the lower-dimensional obstacle problem; and his 2011 paper with N. Matevosyan in *CPAM*, which extends the deep monotonicity formula of Alt–Caffarelli–Friedman and the estimate of Caffarelli–Jerison–Kenig for subsolutions of equations with nonzero right-hand side to the case of elliptic and parabolic equations in divergence form with double Dini coefficients.

In a 2007 joint article with D. Danielli and N. Garofalo in *Adv. in Math.*, Petrosyan took up a new direction with the study of the regularity of the free boundary in the subelliptic obstacle problem.

Arshak Petrosyan received his Ph.D. in 2000 from the Royal Institute of Technology in Stockholm. He then spent three years as a post-doc at the University of Texas at Austin, and joined the Faculty at Purdue University in 2003. He was awarded the 2008–2009 Ruth and Joel Spira Teaching Award for Outstanding Undergraduate Teaching from Purdue University. Jointly with N. Uraltseva and H. Shahgholian, Arshak is in the process of completing a monograph on the study of the regularity of the free boundary in obstacle type problems.

Honors and Awards

Presidential Committee: President Obama has announced that he will nominate Dr. Carlos Castillo-Chavez as a member of the President's Committee on the National Medal of Science. Carlos served on MSRI's Human Resources Advisory Committee from April 1, 1997 to April 1, 2000 and maintains close ties with the Institute.

2011–12 Clay Senior Scholars: Keith Ball, Tobias Colding, William Johnson, Richard Kenyon, Gregory Lawler.

2011–12 UC Berkeley Chancellor's professorship: James Propp

2011–12 Named professorships and fellowships at MSRI:

Fall 2011

Quantitative Geometry

Eisenbud Professorships:
Emmanuel Breuillard
Marianna Csornyei
Tobias Colding
Ursula Hamenstädt
Gideon Schechtman

Simons Professorship:
Assaf Naor
Shmuel Weinberger

Viterbi Postdoctoral Fellowship:
Irene Peng

Spring 2012

Random Spatial Processes

Eisenbud Professorships:
Mireille Bousquet-Mélou
Curtis McMullen
James Propp
Dana Randall
Steffen Rohde
Vladas Sidoravicius
Maria Eulália Vares

Simons Professorship:
Peter Winkler

Viterbi Postdoctoral Fellowship:
Shawn Drenning

For more information on these awards, see the Spring/Summer 2010 Emissary at http://www.msri.org/attachments/media/news/emissary/EmissarySpring2010.pdf.

Math Circles

Dave Auckly

The 2011 **Circle on the Road workshop** took place from March 18 to 20 on the University of Houston campus (see photos). The workshop included a mathematics festival that was open to the public. This is part of our efforts to spread mathematical circles across the United States. The National Association of Math Circles web site (www.mathcircles.org) includes many materials to help people start and run math circles; video from the 2010 Circle on the Road workshop can be found at http://tinyurl.com/2010circles, and lesson plans, some of which include video, at https://www.mathcircles.org/content/lesson-plans.

Four new books will appear in June in the **Mathematical Circles Library**, copublished by the Mathematical Sciences Research Institute and the American Mathematical Society, with the generous support of the John Templeton Foundation:

- *Moscow Mathematical Olympiads, 1993-1999*, edited by R. M. Fedorov, A. Y. Kanel-Belov, A. K. Kovaldzhi, and I. V. Yashchenko;
- *Introduction to Functional Equations and Inequalities*, by Costas Efthimiou;
- *Geometry for Teachers*, by Paul Sally and Judy Sally;
- *Math from Three to Eight*, by Alexander Zvonkin.

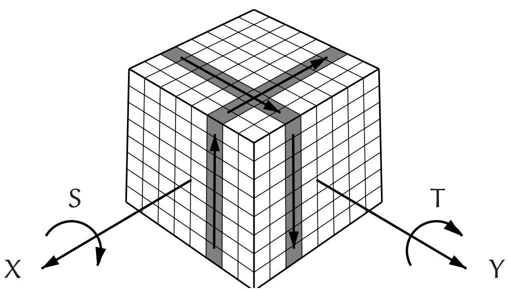
For details see http://library.msri.org/msri-mcl .

Puzzles Column

Joe P. Buhler and Elwyn Berlekamp

1. Consider an $8 \times 8 \times 8$ Rubik's cube. A layer of the cube consists of the cubelets in a fixed plane perpendicular to one of the three axes. For instance, the external layers contain the 64 cubelets on a face of the cube, and an internal layer consists of the 28 cubelets in a ring around the cube.

Let s be the third layer counting inward from a face perpendicular to the x-axis (see figure), and t the third layer counting inward from a face perpendicular to the y-axis. Let S and T be the moves of the cube which rotate the respective slices by 90 degrees as indicated in the figure.



Suppose you apply first T and then S; the combined transformation is called ST. What is the smallest number of repetitions of the transformation ST that can return the cube to its original configuration? (In group theory language: What is the order of ST?)

Comment: This problem appeared on the 2011 Bay Area Mathematical Olympiad, which is associated with the Bay Area Math Circle program that MSRI helps to support.

2. You are arrested and imprisoned in a country with a somewhat unusual judicial system. When you arrive at the prison you are given 100 coins and told that 50 of them are magic coins. To you, the magic coins are indistinguishable from the non-magic coins but your jailers can immediately detect which is which (perhaps the coins have embedded RFID chips...).

Each evening you are allowed to partition all of the coins into two piles (not necessarily of the same size). If each pile contains exactly 25 magic coins, you are released.

Find an algorithm that *guarantees* that you will be released in 50 days.

Comment: This charming problem came to us from Gregory Galperin, via Richard Stong.

3. Suppose that a positive integer n has k divisors, $k \geq 17$, labeled

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Suppose also that $d_7^2 + d_{15}^2 = d_{16}^2$.

(a) Is it possible to have $d_6 > 6$?

(b) What is d_{17}?

Comment: This apparently first appeared in an Olympiad training exam in India, and subsequently in Crux Mathematicorum as well as Stan Wagon's Macalester *Problem of the Week* online column.

4. Let V be the set of monic polynomials of some fixed degree d with coefficients in the field of p elements, for some prime p. Let x be a fixed element of that field.

The probability that a random element f of V has x as a root is $1/p$.

Now let U be the subset of V consisting of squarefree polynomials, i.e., those whose factorization into irreducible polynomials has no repeated factors. What is the probability that a random element f of U has $f(x) = 0$?

Comments: This perhaps surprising result (and generalizations) played an important role in Par Kurlberg's talk at the April 11 workshop at MSRI on Arithmetic Statistics. He also points out that the same idea arises for integers: what is the probability that a random squarefree integer is divisible by a prime p?

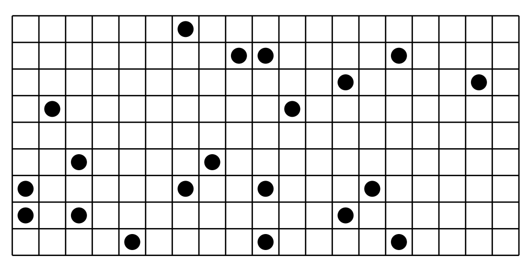
5. A checkerboard with m rows and n columns has markers in k of its squares, whose locations are randomly chosen. Ralph and Cathy must search the squares sequentially in an order of their choosing, and whichever first finds a marked square wins. Ralph chooses to search by row, that is,

$$(1,1),\ (1,2),\ \ldots,\ (m,n-1),\ (m,n),$$

whereas Cathy searches by column, that is,

$$(1,1),\ (2,1),\ \ldots,\ (m-1,n),\ (m,n).$$

If $m < n$, which player is more likely to find a marked square first?



$m = 9$
$n = 19$
$k = 20$

Comment: Tim Chow posed this in the *American Mathematical Monthly* for October 2010, preceded by the more general question of how to compare two arbitrary search sequences (permutations) that Ralph and Cathy might choose.

6. Are there any perfect squares whose decimal expansions contain 1,000,000 digits of which exactly half are even?

Comments: This was brought to our attention by Richard Stong, after Gregory Galperin asked him the harder question of whether there are perfect cubes with the same property. A related question (to which we know one answer) is as follows: specify a million-digit square wherein each of the 10 digits occurs equally often.

What is Arithmetic Statistics?

Brian Conrey, Barry Mazur, Michael Rubinstein, and Nina Snaith

Number Theory has its share of conjecture and heuristics that thrive on — if not depend on — the accumulation of aggregates of instances, aggregates of numerical data.

To see that *numerical data related to numbers themselves* is also at the very heart of the pleasure of number theory, and is a major reason for the very theory itself, consider this letter of Gauss to one of his students (italics mine):

> Even before I had begun my more detailed investigations into higher arithmetic, one of my first projects was to turn my attention to the decreasing frequency of primes, to which end *I counted the primes in several chiliads* [groups of 1000] and recorded the results on the attached white pages. I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm, so that the number of primes below a given bound n is approximately equal to $\int dn/\log n$ [...]. Later on, when I became acquainted with the list in Vega's tables (1796) going up to 400031, I extended my computation further, confirming that estimate. *In* 1811, *the appearance of Chernau's sieve gave me much pleasure and I have frequently (since I lack the patience for a continuous count) spent an idle quarter of an hour to count another chiliad here and there...*

Often, in modern number theory, to actually sample a sufficient quantity of data that might allow you to guess even approximate qualitative behavior of the issue you are studying, you may have to go out to very high numbers. For example, there are basic questions about elliptic curves (e.g., what is the frequency of those possessing two independent rational points of infinite order) where if you only look at curves of conductor $< 10^8$, you might be tempted to make guesses that are not only wrong, but qualitatively wrong.

Our program *Arithmetic Statistics* then stands for those aspects of number theory — be it theory or computation — that connect closely with concrete (important) *numerical data related to numbers themselves*.

Many people in our program are engaged in the theoretical side of our subject, and many in the computational side. Much "theoretical" modern number theory bears on, and sometimes has vital need of large scale computing projects and large databases. And both the computational and theoretical facets connect to some of the famous heuristics in our subject: *Cohen–Lenstra* heuristics (average expected size of various finite abelian groups that appear in our subject); and *random matrix* heuristics. The computational and theoretical facets of our subject form one interlocking unity.

We next discuss a few examples of recent work that have been themes of our program.

• Manjul Bhargava together with his students and coauthors have been developing extremely precise methods for counting appropriate orbits of certain arithmetic groups acting on integral points on certain lattices. This approach follows and significantly refines the classical methods in the geometry of numbers (as had pursued by Gauss, Minkowski, Siegel, and others). A major application of this work of Bhargava and coauthors is to establish counts of important ingredients of the arithmetic of elliptic curves. Among their applications is the result of Bhargava–Shankar that the average rank of the Mordell–Weil group of elliptic curves over $\mathbb{Q}$ — when they are ordered in any of the standard ways — is less than 1.5.

This result is related to their study of the average *size* of the 2-Selmer rank of elliptic curves (again over $\mathbb{Q}$ — and when they are ordered in any of the standard ways). They show that the average size is *three.*[1] For any prime number p the *reduced* p*-Selmer rank of an elliptic curve over a number field*[2] has this important property: it is finite (!), computable (!) (at least in principle), and is an upper bound for the rank of the Mordell–Weil group of the elliptic curve over the number field. If the Shafarevich–Tate conjecture holds, then for all but finitely many primes p, the reduced p-Selmer rank would be equal to that Mordell–Weil rank. So it is natural, as in the results of Bhargava and coauthors alluded to above, to expect that the statistics of p-Selmer ranks (e.g., even when restricted to $p = 2$) contribute to our understanding of Mordell–Weil ranks. In the course of our program we have been learning the most recent advances in this direction (3-Selmer, 5-Selmer).

• The heuristics predicting "average sizes" of quite a few important arithmetic objects have also been the focus of our program. We are fortunate to have both Henri Cohen and Hendrik Lenstra among us. They are the co-originators of the *Cohen–Lenstra heuristics*, which guides conjectures regarding average sizes of ideal class groups and other important invariants in number theory. The latest development in the formidable toolbox of heuristics is due to Bjorn Poonen and Eric Rains and has a somewhat different slant; it gives one precise guesses for the probabilities of reduced p-Selmer ranks for elliptic curves over a given number field (when these curves are ordered in the usual way). This too has been one of the focuses of our program. A few years ago, Peter Swinnerton-Dyer, extending earlier results of Heath-Brown, studied the probabilities of reduced 2-Selmer ranks of families of elliptic curves that are quadratic twists of some very specific types of elliptic curves over $\mathbb{Q}$. One grand (and enticing) feature of Swinnerton-Dyer's study is that the probabilities arise as if they were the product of a specific Markov process; another curious feature — a drawback, perhaps — is that the statistics are dependent upon ordering the elliptic curves in the twist family *not in the standard way* but in terms of the number of primes dividing the discriminant. All the issues that are brought up by this work are focuses of current research in our

[1] Of course no 2-Selmer group can have such a size: these 2-Selmer groups are then all either above or below average.
[2] This is the dimension of the so-called p-Selmer group minus the rank of rational p-torsion of the elliptic curve over the number field.

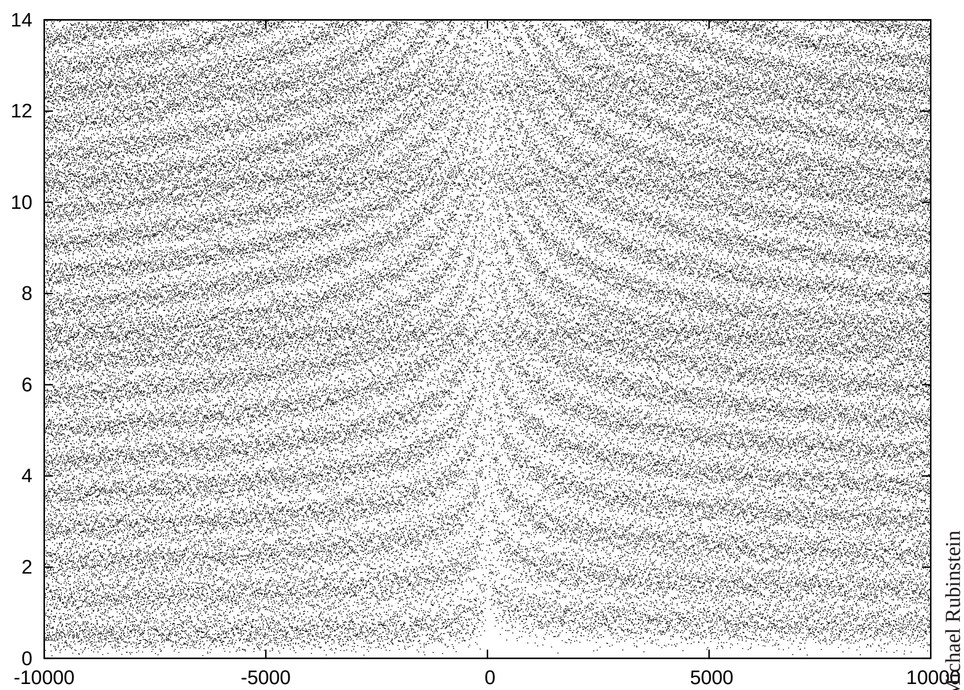
program. Specifically, Dan Kane's work in the program is towards relating such Swinnerton-Dyer statistics dependent upon different orderings of the collection of elliptic curves being sampled, while Karl Rubin, Zev Klagsbrun, and Barry Mazur are developing an approach (which has a "Markov process feel") to unconditionally prove the expected statistics for reduced 2-Selmer ranks over an arbitrary number field for all quadratic twists families of many elliptic curves (the elliptic curves in any of these families are ordered in a certain not entirely unnatural, but again nonstandard, way). In separate work, Jonathan Hanke is collaborating with Bhargava and Shankar to obtain the asymptotics for the 2-part of the class group of n-monogenic orders in cubic fields.

• Dirichlet L-functions are the simplest generalizations of the Riemann zeta function. They have been used to prove an asymptotic formula for the number of primes up to a quantity X in a given arithmetic progression modulo q. Like the Riemann zeta function each Dirichlet L-function can be expressed as Dirichlet series (the Riemann zeta function has Dirichlet series coefficients $1, 1, 1, \ldots$ and the first Dirichlet L-function has coefficients periodic modulo 3, namely $1, -1, 0, 1, -1, 0, \ldots$), has a functional equation and Euler product, and is conjectured to have its zeros on the $\frac{1}{2}$-line; the latter assertion is sometimes called the generalized Riemann hypothesis. It can be proven that each individual Dirichlet L-function has at least 40% of its zeros on the $\frac{1}{2}$-line. Conrey, Iwaniec, and Soundararajan (CIS) have now shown that, when all of the zeros of these Dirichlet L-functions are taken together, at least 55% of these zeros are on the $\frac{1}{2}$-line. To be specific, take a large number Q and consider all of the L-functions associated with a primitive character modulo q where $q \leq Q$. Now consider for all these L-functions all the zeros located in the rectangle of complex numbers with real parts between 0 and 1 and imaginary parts between 0 and $\log Q$. CIS can prove that at least 55% of the zeros in this rectangle have real part $\frac{1}{2}$.

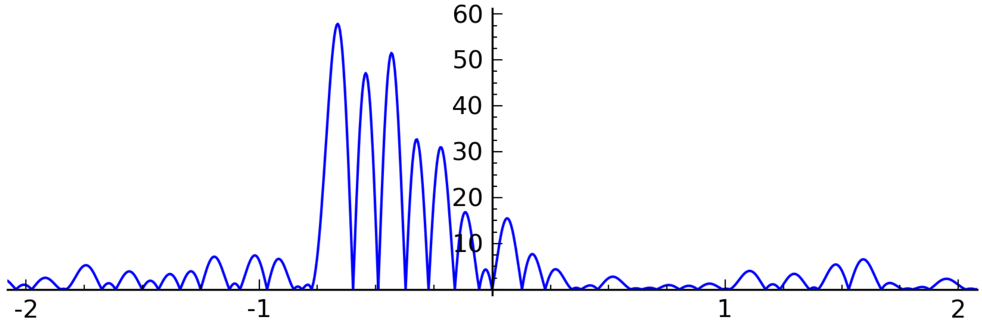
Now, with their asymptotic version of the large sieve, CIS have studied zeros on the $\frac{1}{2}$-line, not only of Dirichlet L-functions, but of other families as well: twists of a fixed L-function of degree 2 by Dirichlet characters (at least 36% of their zeros are on the $\frac{1}{2}$-line) and twists of L-functions of degree 3 (at least 0.5% of their zeros are on the $\frac{1}{2}$-line). In addition, CIS have been able to confirm a prediction from random matrix theory about the sixth moment of Dirichlet L-functions at the point $\frac{1}{2}$, averaged over characters with moduli up to Q. They prove a formula that includes all the main terms and has an error term that is a power of Q smaller than the main terms. The main terms are expressed in terms of simple factors multiplied by a ninth-degree polynomial in $\log Q$. The leading coefficient of the polynomial is 42 and the lower terms are given explicitly in terms of complicated arithmetic and geometric factors. The theorem *exactly* matches the predictions arising from random matrix theory, and provides excellent confirmation of the RMT models for L-functions. We are very fortunate to have as participants in our program all five authors of the paper in which the predictions, now confirmed, were first detailed: Brian Conrey, David Farmer, Jon Keating, Michael Rubinstein, and Nina Snaith.

• Several researchers are examining statistics for curves over finite fields. The zeros of the zeta function are the inverses of the eigenvalues of the Frobenius endomorphism. The work of Katz and Sarnak indicates that when the genus g is fixed and the characteristic q tends to infinity, the normalized zeros are distributed like the eigenvalues of matrices in a group of random matrices determined by the monodromy group of the moduli space of the curves. But the related question of studying statistics as q remains fixed and the genus g grows to infinity is still largely unknown, though recent progress has been made in computing the distribution of the trace of the Frobenius endomorphism for various families by Kurberg–Rudnick, Bucur–David–Feigon–Lalín and Bucur–Kedlaya.

• The technique used by CIS is called the *asymptotic large sieve*. It can be used to give an asymptotic formula for a quantity that would have previously been estimated by the *large sieve inequality*. The latter has been a staple of number theorists for over 40 years. One spectacular application of the large sieve inequality is to prove the Bombieri–Vinogradov theorem, which asserts that when counting primes up to X in arithmetic progressions with moduli up to Q, the error terms behave, on average, as well as could be expected, that is, as well as could be proved assuming the Generalized Riemann Hypothesis. This is a deep and famous result; indeed, Enrico Bombieri won the Fields medal in 1974 for this work. A few years ago Goldston, Pintz, and Yildirim used the BV theorem to prove their celebrated theorem to the effect that the smallest gaps between consecutive prime numbers are an order of magnitude smaller than the average gaps.



The imaginary parts of the zeros of quadratic Dirichlet L-functions of discriminant with absolute value less than 10000.

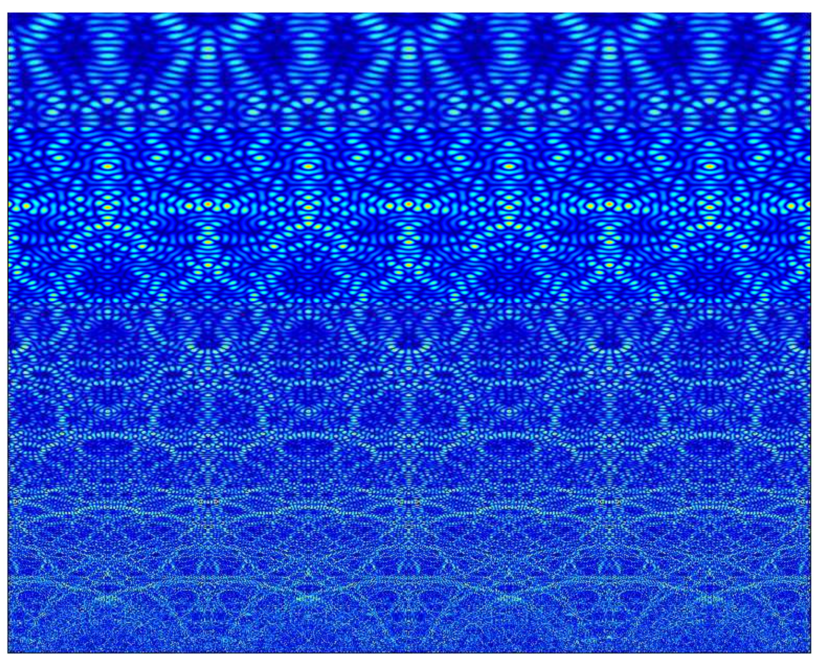


A plot of the zeta function near the 10^{32}nd zero, by Jonathan Bober and Ghaith Hiary.

The broader question of computing the global distribution of the zeros in the g limit remains. This is a nontrivial modeling job, since the global obstruction imposes an infinite, but discrete, set of conditions that the matrix model should satisfy. Such a model needs to exhibit both discrete and continuous features in order to capture the global phenomenon. Bucur and Feigon, together with their collaborators David and Lalín, will be working in this direction while at MSRI.

• Computation and experimentation have been playing a large role in our program. For example, postdocs Jonathan Bober and Ghaith Hiary have been implementing Hiary's world's fastest algorithms for the Riemann zeta function, computing zeros of $\zeta(s)$ with $\operatorname{Im} s$ near 10^{36}, and using their data to test some conjectures about the behavior of the zeta function. Rubinstein is developing general purpose algorithms for computing L-functions and is also gathering extensive numerical evidence in favor of the generalized Riemann hypothesis. William Stein has been tabulating elliptic curves over $\mathbb{Q}(\sqrt{(5)})$, and verifying the Birch–Swinnerton-Dyer conjecture. John Cremona is working on his programs to systematically find curves of a given conductor over $\mathbb{Q}$, with a view to doubling the range of his tables and verifying (or otherwise) that there is no curve of rank 4 and conductor less than 234446. Nathan Ryan, Nils Skoruppa, and Gonzalo Tornaría, in collaboration with Martin Raum, have been studying methods for computing with Siegel modular forms having associated L-functions of degree 4. Duc Khiem Huynh is attempting to develop probabilistic models for these L-functions with the goal of testing the predictions using data provided by Ryan, Skoruppa, Tornaría, and Raum. Skoruppa is working on a new algorithm for computing modular forms of half integral weight directly from the periods of the associated modular forms of integral weight. This will make it possible to tabulate half integral modular forms of very high level without the need of computing complete (and then very high dimensional) spaces as is required by the currently known algorithms. David Farmer, Stefan Lemurell, and Sally Koutsoliotas are developing methods for finding Maass forms for higher rank groups and testing conjectures regarding their Fourier coefficients and associated L-functions. Jonathan Hanke is working with Tornaría, and also collaborator Will Jagy, on classifying regular and spinor regular ternary quadratic forms, improving the modular symbols code in SAGE to make it more suitable for computations proving finiteness theorems, and making tables of quadratic forms in 3 and 4 variables (over $\mathbb{Q}$ and some small number fields) together with Robert Miller.

• In the study of elliptic curves over totally real number fields like $\mathbb{Q}(\sqrt{5})$ (recent work of William Stein) one is naturally led to Hilbert modular forms. Work of Shimura and recently of Ikeda in Japan indicates that there is a similar connection between modular forms of half integral weight and modular forms of integral weight over number fields as it is well-known for $\mathbb{Q}$. However, as it is known from the theory over $\mathbb{Q}$, there are several advantages to replace in such a theory the modular forms of half integral weight by Jacobi forms. The Fourier coefficients of these Jacobi forms correspond (in the theory over $\mathbb{Q}$) to the central value in the critical strip of the twisted L-series of the associated Hilbert modular forms or elliptic curve over $\mathbb{Q}$. Skoruppa and his student Hatice Boylan are preparing a longer article to set up such a theory over arbitrary number fields based on results of Boylan's thesis. In particular, they want, in joint work with Fredrik Strömberg, to compute sufficiently many examples of Jacobi forms over $\mathbb{Q}(\sqrt{5})$ which should complement the computations of William Stein on elliptic curves over $\mathbb{Q}(\sqrt{5})$.



Contour plot of a Maass form, by Fredrik Strömberg. Another example can be seen on page 1.

In the case of Siegel modular forms a conjecture of Böcherer — originally stated in the case of the forms for the full symplectic group — relates sums of Fourier coefficients of a form to the central values of its twisted spinor L-series, generalizing the formulas for the coefficients of Jacobi forms (over $\mathbb{Q}$) mentioned above. Similar formulas for the case of paramodular forms are being investigated by Ryan and Tornaría. This case is of particular interest due to the so-called Paramodular Conjecture, which proposes a bridge to geometry by relating spinor L-series attached to paramodular forms with Hasse–Weil L-functions attached to rational abelian surfaces (analogue to the Modularity Theorem of Wiles et al.). Ryan and Tornaría are trying to find algorithms to compute Fourier coefficients of these paramodular forms on a large scale. This would not only provide more evidence for both the Paramodular Conjecture and the paramodular extension of Böcherer Conjecture, but would also allow a large scale computation of central values of twists for degree 4 L-series, useful for testing and refining random matrix heuristics for degree 4 L-series.

How the Program Unfolded

Learning seminars, whereby our participants meet weekly to teach each other and discuss material relevant to our research, form an important part of our program. The Bhargava–Shankar group has been meeting to learn material related to the work of Bhargava and Shankar on ranks of elliptic curves. The explicit formula group is studying the problem of ranks from an analytic perspective. The low lying zeros seminar has been looking at papers related to the distribution of zeros in families of L-functions. Quadratic twists of elliptic curves meets to discuss the problem of ranks of elliptic curves in families of quadratic twists. Earlier, a group was meeting to study the Cohen–Lenstra heuristics and its extension to Tate–Shafarevich groups by Christophe Delaunay. Lastly, a few researchers are holding a seminar to study paramodular forms.

The first workshop to take place as part of the Arithmetic Statistics program was the two-day Connections for Women event. This targeted female mathematicians in fields related to the program, but we were pleased to see that all aspects of the workshop were well-attended by the program's participants, which lead to a very even mix of male and female researchers. The Connections for Women workshop was a very agreeable mixture of excellent talks, a buzz of mathematical discussion and a chance to meet new people; every math workshop should be like this! The audience enjoyed six superb talks by leading women in the area, ranging from the number theory involved in cryptography to several of the questions of counting (ranks, points on curves, number fields) that are themes of the rest of the program.

The discussion session on pursuing a career in mathematics saw senior mathematicians giving advice on how to apply for first jobs and postdoctoral positions, some anecdotes about how dual-career couples have found posts in the same institution, and strategies for departments keen to increase the number of women in their faculty. With participants covering the spectrum from undergraduates to those with a long career behind them, the discussion was lively and productive.

These two days then lead into the main Introductory Workshop for the Arithmetic Statistics program, which most of the Connections participants stayed on to enjoy.

Three other workshops form a part of our program. Our introductory workshop was held from January 31 to February 4 and featured talks to help define the direction of our program. Talks were given, in order of appearance, by: Henri Cohen, Karl Rubin, Manjul Bhargava, Michael Rubinstein, Nina Snaith, Melanie Wood, Brian Conrey, Andrew Sutherland, Jordan Ellenberg, David Farmer, John Voight, Henryk Iwaniec, Akshay Venkatesh, John Cremona, Bjorn Poonen, William Stein, Kannan Soundararajan, Chantal David, and Frank Thorne.

Several of the participants in our program are also involved in a large scale NSF-funded collaborative Focused Research Group project to develop methods for computing with L-functions and associated automorphic forms, as well as verify many of the important conjectures in this area. To help diffuse the large amount of data being generated by the project, an archive with a user-friendly front end for browsing and searching the data is being developed, and a workshop involving fifteen participants was held at MSRI from February 21 to 25 to continue developing the archive.

The last workshop for our program was held from April 11 to 15 on the theme of Arithmetic Statistics. It highlighted some of the work being carried out at MSRI during our program.

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