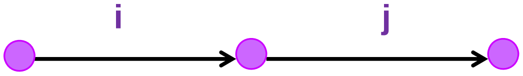
The Edge Matrix W_1

Define W_1 to be the $2|E|\times 2|E|$ matrix with i j entry 1 if edge i feeds into edge j, (end vertex of i is start vertex of j) provided i ≠ opposite of j, otherwise the i j entry is 0.

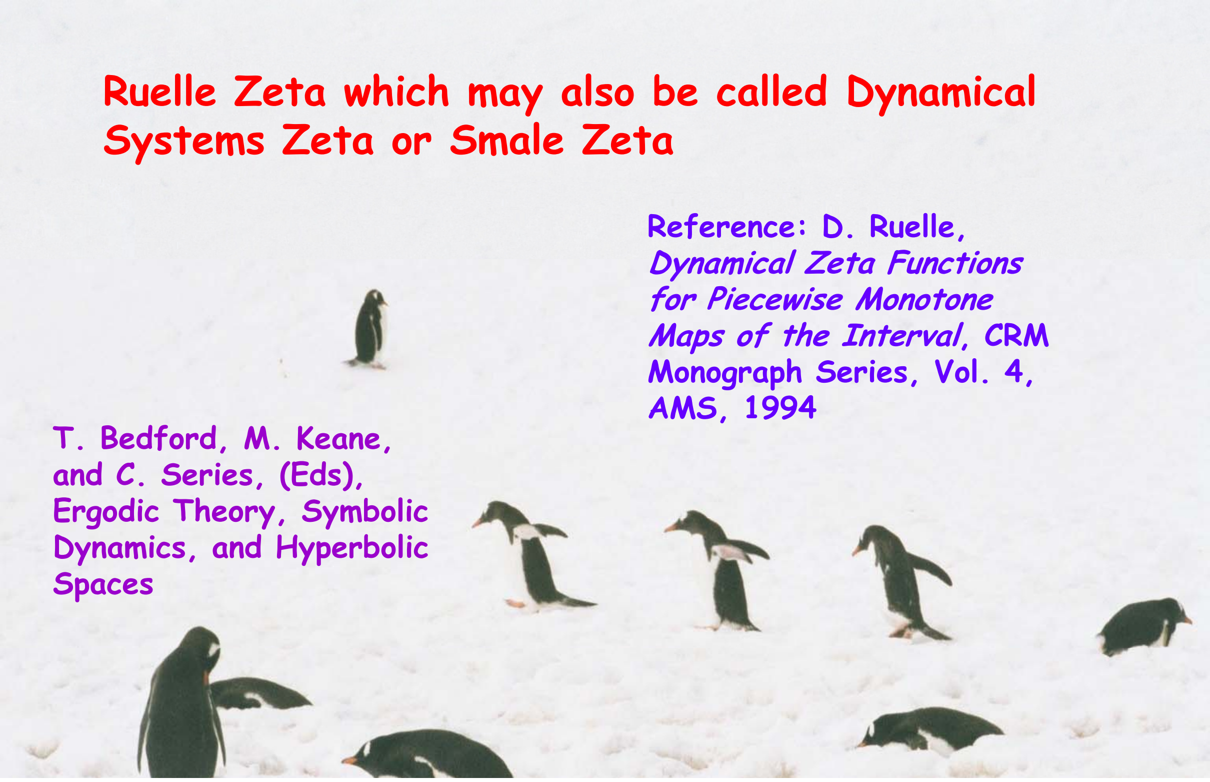


Theorem. $\zeta(u,X)^{-1}=\det(I- W_1 u)$.

Corollary. The poles of Ihara zeta are the reciprocals of the eigenvalues of W_1. Recall that R = radius of convergence of Dirichlet series for Ihara zeta.

Note: R is closest pole of zeta to 0.

The pole R of zeta is: R=1/Perron-Frobenius eigenvalue of W_1. See Horn & Johnson, *Matrix Analysis*



Ruelle Zeta which may also be called Dynamical Systems Zeta or Smale Zeta

Reference: D. Ruelle, *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval*, CRM Monograph Series, Vol. 4, AMS, 1994

T. Bedford, M. Keane, and C. Series, (Eds), Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces

Ruelle's motivation for his definition came partially from Artin and Mazur, *Annals of Math.*, 81 (1965).

They based their zeta on the zeta function of a projective non-singular algebraic variety V of dimension n over a finite field k with q elements.

If N_m is the number of points of V with coordinates in the degree m extension field of k, the zeta function of V is:

$$Z_V(z) = \exp\left(\sum_{m\geq 1} \frac{N_m z^m}{m}\right).$$

$N_m=|Fix(F^m)|$, where F is the Frobenius map taking a point with coordinates x_i to the point with coordinates $(x_i)^q$.

Weil conjectures, proved by Deligne, say

$$Z_V(z) = \prod_{j=0}^{2n} P_j(z)^{(-1)^{j+1}},$$

where the P_j are polynomials with zeros of absolute value $q^{-j/2}$. Moreover the P_j have a cohomological meaning as $\det(1-zF^*|H^j(V))$.

Artin and Mazur replace the Frobenius of V with a diffeomorphism f of a smooth compact manifold M - defining their zeta function

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} \left|Fix(f^m)\right|\right).$$

Ruelle zeta function

Suppose M is a compact manifold and f:M→M.
Assume the following set finite: $\mathrm{Fix}(f^m)=\{x\in M \mid f^m(x)=x\}$.
1st type Ruelle zeta is defined for matrix-valued function $\varphi: M\to\mathbb{C}^{d\times d}$

$$\zeta(s)=\exp\left\{\sum_{m\geq 1}\frac{z^m}{m}\sum_{x\in Fix(f^m)} Tr\left(\prod_{k=0}^{m-1}\varphi\left(f^k(x)\right)\right)\right\}$$

A special case: $\varphi=1$

$$\zeta(z)=\exp\left(\sum_{m=1}^{\infty}\frac{z^m}{m}\left|Fix(f^m)\right|\right).$$

I=finite non-empty set (our alphabet). For a graph X, I is the set of directed edges.
The transition matrix t is a matrix of 0's and 1's with indices in I. In the case of a graph X, t is the 0,1 edge matrix W_1 defined earlier, which has i,j entry 1 if edge i feeds into edge j (meaning that terminal vertex of I is the initial vertex of j) provided edge i is not the inverse of edge j.

Note: $I^{\mathbb{Z}}$ is compact and so is the closed subset

$$\Lambda=\left\{(\xi_k)_{k\in\mathbb{Z}}\,\middle|\, t_{\xi_k\xi_{k+1}}=1,\forall k\right\}.$$

In the graph case $\xi\in\Lambda$ corresponds to a path without backtracking.
A continuous function $\tau:\Lambda\to\Lambda$ such that $\tau(\xi)_k=\xi_{k+1}$ is called a subshift of finite type.
Prop. 1. (Bowen & Lanford). The Ihara zeta is the reciprocal of a polynomial:

$$\zeta(z)=\exp\left(\sum_{m\geq 1}\frac{z^m}{m}Tr(t^m)\right)=\det(I-zt)^{-1}.$$

Proof.
By the first exercise below, we have the first equality in the theorem. Then $|Fix(\tau^m)|=Tr(t^m)$. This implies using the 2nd exercise:

$$\zeta(z)=\exp(Tr(-\log(1-zt)))=\det(I-zt)^{-1}.$$

Exercise A. Show that in the graph case, $|Fix(\tau^m)|$=the number of length m closed paths without backtracking or tails in the graph X with $t=W_1$, from our previous discussion of graphs.

Exercise B. Show that $\exp(TrA)=\det(\exp A)$ for any matrix A. Hint: Use the fact that there is a non-singular matrix B such that $BAB^{-1}=T$ is upper triangular.

Define R as the radius of convergence of the Ihara zeta. It is also the closest pole to 0. For a (q+1)-regular graph, $R=1/q$.

Define Δ as the g.c.d. of the prime lengths.

Theorem. (Graph prime number theorem)
If the graph is connected and m divides Δ,

$$\pi(m) = \#\{\text{prime paths of length } m\} \sim \Delta R^{-m}/m, \quad \text{as } m \to \infty$$

Proof.

If N_m=# {closed paths C,length m,no backtrack,no tails}

we have
$$u\frac{d\log\zeta(u,X)}{du}=\sum_{m=1}^{\infty}N_m u^m$$

Then $\zeta(u,X)^{-1}=\det(I-W_1 u)$ **implies**

$$u\frac{d\log\zeta(u)}{du}=\sum_{n\geq1}N_m u^m=-\sum_{\lambda\in Spec(W_1)}\log(1-\lambda u)$$
$$=\sum_{\lambda\in Spec(W_1)}\sum_{m\geq1}\frac{(\lambda u)^m}{m}=\sum_{m\geq1}\left(\sum_{\lambda\in Spec(W_1)}\frac{\lambda^m}{m}\right)u^m$$

$$N_m=\sum_{\lambda\in Spec(W_1)}\frac{\lambda^m}{m}$$

The dominant terms in this sum are those coming from the eigenvalues λ of W_1 with $|\lambda|=1/R$.

$$\sum \quad \lambda$$

Next we need a formula to relate N and π

$$\zeta(u,X)=\prod_{n\geq 1}\left(1-u^{n}\right)^{-\pi(n)}$$

This implies

$$u\frac{d\log\zeta(u,X)}{du}=\sum_{m\geq 1}\sum_{d|m}d\pi(d)u^{m}$$

$$N_m=\sum_{d|m}d\pi(d)$$

Möbius inversion implies

$$\pi(m)=\frac{1}{m}\sum_{d|m}\mu(\frac{m}{d})N_d$$

Note that $\pi(1)=0$ so that when m is prime p, we have $N_p=\pi(p)$.

Thus the prime number theorem follows. Exercise. Fill in the details in this proof.

Tetrahedron example

$$x\frac{d}{dx}\log\zeta_X(x)=\sum_{m\geq 1}N_m x^m$$

$$= 24x^3 + 24x^4 + 96x^6 + 168x^7 + 168x^8 + 528x^9 + 1200x^{10} + 1848x^{11} + 3960x^{12} + 8736x^{13} + 16128x^{14} + 31944x^{15} + 66888x^{16}$$

⇒ 8 prime paths of length 3 on the tetrahedron. Check it! We count 4 plus their inverses to get 8.

⇒ 6 paths of length 4. We agree.

⇒ 0 paths of length 5

⇒ 16 paths of length 6. That is harder to check.

Question: 528 is not divisible by 9. Shouldn't it be?

Answer: You only know m divides N_m when m is prime.

$$\pi(m)=\frac{1}{m}\sum_{d|m}\mu(\frac{m}{d})N_d$$

Exercises

1. List all the zeta functions you can and what they are good for. There is a website that claims to list lots of them:
   www.maths.ex.ac.uk/~mwatkins
2. Compute Ihara zeta functions for the cube, dodecahedron, buckyball, your favorite graph. Mathematica should help.
3. Do the basic graph theory exercise 14 on page 24 of the manuscript on my website:
   www.math.ucsd.edu/~aterras/newbook.pdf
4. Show that the radius of convergence of the Ihara zeta of a (q+1)-regular graph is R=1/q. Explain why the closest pole of zeta to the origin is at R.
5. Prove the functional equations of Ihara zeta for a regular graph. See p. 25 of my manuscript.
6. Look up the paper of Kotani and Sunada and figure out their proof. You need the Perron – Frobenius theorem from linear algebra. [Zeta functions of graphs, J. Math. Soc. Univ. Tokyo, 7 (2000)].

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6. Fill in the details in the proof of the graph theory prime number theorem.
7. Prove the prime number theorem for a (q+1)-regular graph using Ihara's theorem with its 3-term determinant rather than the $1/\det(I-uW_1)$ formula.
8. Exercise 16 on page 28 of my manuscript. This is a Mathematica exercise to plot poles of Ihara zetas.

9. Show that in the graph case, $|\mathrm{Fix}(\tau^m)|$=the number of length m closed paths without backtracking or tails in the graph X with $t=W_1$, from our previous discussion of graphs.
10. Show that exp(TrA)=det(expA) for any matrix A. Hint: Use the fact that there is a non-singular matrix B such that $BAB^{-1}=T$ is upper triangular.